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TECHNICAL MEMORANDUM

No. 1063

GAS JETS

By S. Chaplygin

Scientific Memoirs. Moscow University, 1902.

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By S. Chaplygin

INTRODUCTORY REMARKS

In his memoir Helmholtz (reference 1) showed the possibility of mathematical analysis of those types of flow of incompressible liquids that are characterized by the formation of so-called rays (Strahle) or jets within the region at rest. Following the work of Helmholtz a rather large number of investigations, devoted to the same problem, appeared in foreign and Russian scientific literature. At present the fully worked out Joukowsky method (reference 2) permits the solution of any problem on steady, irrotational flow of an ideal liquid under the following conditions: first, the fluid throughout moves parallel to a certain plane, the flow being bounded by plane walls perpendicular to this plane, and secondly, the motion takes place in the absence of external forces. (The same conditions are imposed in almost all problems of this type.)

The analogous problem for an ideal gas has hardly been touched upon. The author is familiar with only one paper which deals with gas jets; namely, the one by P. Molenbroek (reference 3). Molenbroek set up the differential equations on which the problem of gas jet flow depends and gave certain particular integrals of these equations; these equations, however, hardly correspond even to the theoretically conceived motion of the gas.

In the present paper a method is presented with the aid of which it is possible, in many cases, to find the solution of a given problem on the flow of an ideal gas.

*Scientific Memoirs, Moscow University, 1902, pp. 1-121.

Here it is necessary to impose all the conditions which are assumed in the analogous problems on an incompressible liquid as previously mentioned by Joukowsky (reference 2). But, in addition, the applicability of the analysis here developed is further restricted by the special requirement that the velocity of the gas particles must nowhere exceed the velocity of sound for the particular physical state of the gas at a given point (local velocity of sound). Corresponding restrictions are likewise imposed on the limits within which the pressure may vary. If this additional condition is not satisfied, stable motion apparently is not possible. It is assumed, however, that with the aid of a certain hypothesis, stated in this paper, the problem can be analyzed also for the case where the additional condition is not satisfied. The mathematical treatment of this problem, however, is left to another paper.

A brief summary of the contents of this paper is presented here.

In part I the differential equations of the problem of a gas flow in two dimensions ~~is~~ are derived and the particular integrals by which the problem on jets is solved are given. Use is made of the same independent variables as Molenbroek used, but it is found to be more suitable to consider other functions. The stream function and velocity potential corresponding to the problem are given in the form of series.

The investigation of the convergence of these series in connection with certain properties of the functions entering them forms the subject of part II.

In part III the problem of the outflow of a gas from an infinite vessel with plane walls is solved.

In part IV the impact of a gas jet on a plate is considered and the limiting case where the jet expands to infinity changing into a gas flow is taken up in more detail. This also solved the equivalent problem of the resistance of a gaseous medium to the motion of a plate.

Finally, in part V, an approximate method is presented that permits a simpler solution of the problem of jet flows in the case where the velocities of the gas (velocities of the particles in the gas) are not very large.

A number of supplementary notes are appended at the end of this report, the second of which establishes a relation between the analysis of part V with certain problems in the theory of minimal surfaces.

A further interesting remark may be noted here: The results obtained in parts III and IV, at least qualitatively, agree sufficiently well with test results, although the experimental investigation of the phenomena accompanying the jet formation was conducted under conditions very different from those assumed in these theoretical investigations.

The principles of the method with its application to flows presented here were briefly communicated to the Moscow Mathematical Society at the beginning of 1896. A more detailed presentation was made at the eleventh session of the experimental scientists and doctors in 1901.

In concluding these introductory remarks deep appreciation is expressed to E. A. Bolotov for his kind help in proofreading the manuscript.

PART I

GENERAL PRINCIPLES OF THE METHOD OF INVESTIGATION

An infinite mass of a perfect gas contained between two parallel planes is assumed and, in addition, bounded by certain cylindrical surfaces perpendicular to these planes. One of the latter is assumed as the coordinate plane XY. Let the gas be in stabilized motion and let the direction of the velocity throughout be parallel to XY. The effect of external forces will be neglected and it will be assumed that the velocities have a potential. Since it is desirable to avoid vorticity formation, it is necessary to consider the pressure as a function of the density. It is convenient to take

$$p = k\rho^{\gamma} \tag{1}$$

and thus assume an adiabatic process.

The magnitude γ equal, for atmospheric air, to 1.4025 (reference 4) is the ratio of the specific heats. It is preferred to consider the motion as constant heat process in view of the small heat conductivity and radiation of the gas particles. Because of this the adiabatic process at large velocities appears most closely approaching the true conditions. In any case, the result of this analysis must be considered as a first approximation for the reason that no account is taken of the connecting chains between the particles and the resulting viscosity forces, friction at the walls, and so forth, factors which, in the case of gas flows, are possibly of greater effect than in the case of liquid flows.

Under the foregoing assumptions the velocity potential φ is a function of x and y and, for the components of the velocity u, v, the expressions

$$u = \frac{\partial\varphi}{\partial x}, \quad v = \frac{\partial\phi}{\partial y} \tag{2}$$

With the density of the gas denoted by ρ the condition of continuity is written

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{3}$$

The Bernoulli law in this problem may, with the aid of equation (1), be reduced to the relation

$$\rho = \rho_o \left(1 - \frac{V^2}{2a}\right)^{\beta\gamma} \tag{4}$$

? cf. eq.(5)

where

$$V^2 = u^2 + v^2, \quad a = \frac{k\gamma\rho_o^{\gamma-1}}{\gamma-1}, \quad \beta = \frac{1}{\gamma-1}, \quad \rho_o = \text{Constant} \tag{4'}$$

the density at the point of the gas is evidently ρ_o where $V = 0$.

For briefness,

$$\frac{V^2}{2a} = \tau$$

so that

$$\rho = \rho_o (1-\tau)^{\beta} \tag{5}$$

Equation (3) indicates the existence of a function ψ determined by the equations

$$\frac{\rho}{\rho_o} u = \frac{\partial \psi}{\partial y}, \quad \frac{\rho}{\rho_o} v = -\frac{\partial \psi}{\partial x} \tag{6}$$

From equations (2) and (6) with the aid of equation (5) a relation is obtained between the functions φ and ψ given by the formulas

$$\left.\begin{aligned} \frac{\partial\psi}{\partial y} &= (1-\tau)^{\beta}\,\frac{\partial\varphi}{\partial x} \\ \frac{\partial\psi}{\partial x} &= -\,(1-\tau)^{\beta}\,\frac{\partial\varphi}{\partial y} \end{aligned}\right\} \qquad (7)$$

The function ψ represents the stream function, the equation ψ = Constant being the equation of a streamline. By assigning successive constant values C_1 and C_2 to the latter, it is readily shown that $(C_1 - C_2)\rho_o$ expresses the mass of gas per second flowing through a cross section of the jet between the streamlines.

$$\psi = C_1, \qquad \psi = C_2$$

Equations (7) are transformed by taking φ, ψ, for the independent variables and considering x, y as functions of φ and ψ. The relations

$$D\,\frac{\partial x}{\partial\varphi} = \frac{\partial\psi}{\partial y}, \qquad D\,\frac{\partial y}{\partial\varphi} = -\,\frac{\partial\psi}{\partial x}$$

$$D\,\frac{\partial x}{\partial\psi} = -\,\frac{\partial\varphi}{\partial y}, \qquad D\,\frac{\partial y}{\partial\psi} = \frac{\partial\varphi}{\partial x}$$

are readily obtained where

$$D = \frac{\partial\varphi}{\partial x}\,\frac{\partial\psi}{\partial y} - \frac{\partial\varphi}{\partial y}\,\frac{\partial\psi}{\partial x} = (1-\tau)^{\beta}\,V^2$$

From the foregoing equations the reciprocal of the square of the velocity is obtained:

$$\frac{1}{V^2} = \left(\frac{\partial x}{\partial\varphi}\right)^2 + \left(\frac{\partial y}{\partial\varphi}\right)^2 \qquad (7')$$

Equations (7) become

$$(1-\tau)^{\beta}\frac{\partial y}{\partial\psi}=\frac{\partial x}{\partial\varphi};\ (1-\tau)^{\beta}\frac{\partial x}{\partial\psi}=-\frac{\partial y}{\partial\varphi}. \tag{8}$$

It is necessary to consider the derivatives with respect to φ and ψ of the variables $\tau = V^2/2\alpha$ and

$$\theta=\operatorname{arctg}\frac{\frac{\partial\varphi}{\partial y}}{\frac{\partial\varphi}{\partial x}}=\operatorname{arctg}\frac{\frac{\partial y}{\partial\varphi}}{\frac{\partial x}{\partial\varphi}}=-\operatorname{arctg}\frac{\frac{\partial x}{\partial\psi}}{\frac{\partial y}{\partial\psi}} \tag{8'}$$;

the inclination of the velocity to the X = axis is evidently θ. Differentiation of V^2 with respect to ψ results in

$$\frac{\partial(V^2)}{\partial\psi}=-2V^4\left(\frac{\partial x}{\partial\varphi}\ \frac{\partial^2 x}{\partial\varphi\partial\psi}+\frac{\partial y}{\partial\varphi}\ \frac{\partial^2 y}{\partial\varphi\partial\psi}\right),$$

or, on the basis of equation (8)

$$\frac{\partial(V^2)}{\partial\psi}=-2V^4\left(\frac{\partial y}{\partial\psi}\ \frac{\partial^2 x}{\partial\varphi\partial\psi}-\frac{\partial x}{\partial\psi}\ \frac{\partial^2 y}{\partial\varphi\partial\psi}\right)(1-\tau)^{\beta};$$

Differentiation of θ with respect to φ gives

$$\frac{\partial\theta}{\partial\varphi}=-\frac{\partial}{\partial\varphi}\operatorname{arctg}\frac{\frac{\partial x}{\partial\psi}}{\frac{\partial y}{\partial\psi}}=-\frac{\frac{\partial y}{\partial\psi}\ \frac{\partial^2 x}{\partial\varphi\partial\psi}-\frac{\partial x}{\partial\psi}\ \frac{\partial^2 y}{\partial\varphi\partial\psi}}{\left(\frac{\partial x}{\partial\psi}\right)^2+\left(\frac{\partial y}{\partial\psi}\right)^2},$$

or

$$\frac{\partial\theta}{\partial\varphi}=-V^2(1-\tau)^{2\beta}\left(\frac{\partial y}{\partial\psi}\ \frac{\partial^2 x}{\partial\varphi\partial\psi}-\frac{\partial x}{\partial\psi}\ \frac{\partial^2 y}{\partial\varphi\partial\psi}\right);$$

These relations lead to the equation

$$\frac{\partial \lg V^2}{\partial\psi}=2(1-\tau)^{-\beta}\frac{\partial\theta}{\partial\varphi}. \tag{9}$$

Further

$$\frac{\partial\theta}{\partial\psi}=\frac{\partial}{\partial\psi}\operatorname{arctg}\frac{\frac{\partial y}{\partial\varphi}}{\frac{\partial x}{\partial\varphi}}=V^2\left(\frac{\partial x}{\partial\varphi}\ \frac{\partial^2 y}{\partial\varphi\partial\psi}-\frac{\partial y}{\partial\varphi}\ \frac{\partial^2 x}{\partial\varphi\partial\psi}\right);$$

whence, with the aid of equation (8), there is obtained

$$\frac{\partial\theta}{\partial\psi}=V^2(1-\tau)^{\beta}\left(\frac{\partial y}{\partial\psi}\ \frac{\partial^2 y}{\partial\varphi\partial\psi}+\frac{\partial x}{\partial\psi}\ \frac{\partial^2 x}{\partial\varphi\partial\psi}\right);$$

But

$$\frac{\partial y}{\partial \psi}\frac{\partial^2 y}{\partial \varphi \partial \psi} + \frac{\partial x}{\partial \psi}\frac{\partial^2 x}{\partial \varphi \partial \psi} = \frac{1}{2}\frac{\partial}{\partial \varphi}\left\{\left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2\right\}$$

$$= \frac{1}{2}\frac{\partial}{\partial \varphi}\left\{\frac{1}{V^2}(1-\tau)^{-2\beta}\right\} = \frac{1}{4\alpha}\frac{\partial}{\partial \varphi}\frac{(1-\tau)^{-2\beta}}{\tau}$$

$$= -\frac{1}{4\alpha\tau^2}[1-(2\beta+1)\tau](1-\tau)^{-2\beta-1}\frac{\partial \tau}{\partial \varphi} .$$

By making the final reduction in the formula for $\frac{\partial \theta}{\partial \psi}$ and substituting τ for V^2 in relation (9) gives the relations:

$$\left.\begin{aligned} \frac{\partial \tau}{\partial \psi} &= 2\tau(1-\tau)^{-\beta}\frac{\partial \theta}{\partial \varphi} \\ \frac{\partial \tau}{\partial \varphi} &= -2\tau\frac{(1-\tau)^{\beta+1}}{1-(2\beta+1)\tau}\frac{\partial \theta}{\partial \psi} \end{aligned}\right. \qquad (10)$$

Passing next to the independent variables τ, θ and taking φ and ψ as functions of the former gives the formulas required:

$$\left.\begin{aligned} \frac{\partial \varphi}{\partial \theta} &= 2\tau(1-\tau)^{-\beta}\frac{\partial \psi}{\partial \tau} \\ \frac{\partial \varphi}{\partial \tau} &= -\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)}(1-\tau)^{-\beta}\frac{\partial \psi}{\partial \theta} \end{aligned}\right\} \qquad (11)$$

$$\frac{\partial}{\partial \tau}\left\{2\tau(1-\tau)^{-\beta}\frac{\partial \psi}{\partial \tau}\right\} + \frac{1-(2\beta+1)\tau}{2\tau(1-\tau)}(1-\tau)^{-\beta}\frac{\partial^2 \psi}{\partial \theta^2} = 0 \qquad (12)$$

Equations (11) and (12) constitute a solution of the problem of the flow of a gas if the range of variables τ,

θ corresponding to the flow is known, if the values of ψ on the boundary streamlines are given, if everywhere within the plane τ, θ the function ψ, together with its first derivatives, is finite, single-valued, and continuous and the magnitude τ does not exceed $1/(2\beta+1)$ and becomes zero only at certain points of the contour. The τ, θ region will be considered singly connected and closed.

In order to show that the function ψ is fully defined for the given conditions it will be proved that the contrary is not true. Let it be assumed that there exist two functions ψ_1 and ψ_2 satisfying all these conditions. It will be shown that $\psi_1 - \psi_2 = 0$. The function $\psi_3 = \psi_1 - \psi_2$ everywhere in the given region of values τ, θ is finite and continuous, satisfies equation (12), and at the boundary of the region assumes the value zero. Multiply the left side of equation (12) by $\psi d\tau d\theta$ and integrate within the limits of the τ, θ region. If the result of the integration is denoted by I, substituting ψ_3 for ψ there is readily obtained:

$$I = -\iint \left\{2\tau(1-\tau)^{-\beta}\left(\frac{\partial\psi_3}{\partial\tau}\right)^2 + \frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}\left(\frac{\partial\psi_3}{\partial\theta}\right)^2\right\}d\tau d\theta$$

$$+\int\left\{2\tau(1-\tau)^{-\beta}\psi_3\frac{\partial\psi_3}{\partial\tau}d\theta + \frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}}\psi_3\frac{\partial\psi_3}{\partial\theta}d\tau\right\} = 0$$

where the double integral extends over the entire τ, θ region, once over its contour. Since on the contour $\psi_3 = 0$ the equation $I = 0$ can be true only if the double integral becomes zero. Under the above-indicated conditions, however, the function under the integral sign may be either positive or zero. It is clear that the zero value must be taken, and this leads to the equations

$$\frac{\partial\psi_3}{\partial\tau} = 0, \qquad \frac{\partial\psi_3}{\partial\theta} = 0 \quad \text{and} \quad \psi_3 = \text{constant} = 0$$

as was required to be proved.

Among the required conditions for the existence of a definite solution it has been mentioned that throughout the region of gas flow the inequality

$$\tau < \frac{1}{2\beta + 1}$$

must be satisfied. The significance of this requirement will be explained. Turning to formulas (4) and (4') for

$$\tau = \frac{V^2}{2\alpha} \leqq \frac{1}{2\beta+1} = 0.17 \qquad (13)$$

gives

$$V^2 \leqq \frac{2\alpha}{2\beta+1}; \qquad V^2 \leqq \frac{2k\gamma\rho_0{}^{\gamma-1}}{1+\gamma};$$

where

$$V^2 = \frac{2k\gamma\rho_0{}^{\gamma-1}}{1+\gamma}, \quad \tau = \frac{1}{2\beta+1}, \quad \rho^{\gamma-1} = \rho_0{}^{\gamma-1}(1-\tau), \quad \rho^{\gamma-1} = \frac{2\rho_0{}^{\gamma-1}}{1+\gamma}$$

whence

$$V^2 = k\gamma\rho^{\gamma-1}$$

or, making use of relation (1) gives

$$V^2 = \frac{p}{\rho}\gamma \qquad (13')$$

Thus the restriction imposed on τ is equivalent to the requirement that the velocity of the gas nowhere exceed the velocity of propagation of sound for the particular physical conditions at the point under consideration. It is supposed that such velocities, at least for established flows, cannot even exist. (See also reference 5, and the authors cited by him.)

The limiting value $\tau = 1/(2\beta+1)$ establishes also the limits within which the pressure may vary in the region occupied by the moving gas mass. Thus, if the variable τ is everywhere less than its limiting value, then

$$\rho \geqq \rho_0 \left(1 - \frac{1}{2\beta+1}\right)^{\beta}, \qquad p \geqq p_0 \left(1 - \frac{1}{2\beta+1}\right)^{\beta\gamma}$$

But $\beta\gamma = \gamma/(\gamma - 1) = 1 + \beta$; hence

$$\frac{p}{p_0} \geqq \left(\frac{2\beta}{2\beta+1}\right)^{\beta+1} = \left(\frac{5}{6}\right)^{7/2} = 0.53, \quad \frac{p_0}{p} \leqq 1.89$$

if it is assumed that γ equals approximately 1.40.

The author turns to the derivation of other very important theorems with regard to the motion under consideration to show, in the first place, that <u>the velocity potential φ, considered as a function of the coordinates, can nowhere, within the flow region, have either a maximum or a minimum</u>. To prove this, it might be possible to consider only the following condition. If a point existed at which φ had a maximum, there would then have to exist about it a closed curve on which φ had a constant value less than the maximum. In such case the gas would flow through this curve from outside to inside the area bounded by it. The mass, bounded by the curve would increase with time and the motion could not be steady. By similar consideration the assumption of a minimum of φ is likewise shown to be impossible. But since <u>the theorem on the function φ holds also for the function ψ, and in view of the fact that it is true also for the coordinates x, y, regarded as functions of the independent variables φ and ψ</u>, another proof applicable to all these functions also will be given.

From formulas (7),

$$\frac{\partial}{\partial x}(1-\tau)^{\beta}\frac{\partial\varphi}{\partial x} + \frac{\partial}{\partial y}(1-\tau)^{\beta}\frac{\partial\varphi}{\partial y} = 0$$

At a certain point A of the flow region let φ have a maximum (or minimum). About this point take a closed curve (C) along which φ maintains a constant value k, less than the maximum (or greater than the minimum). Multiply the above equation in φ by $F(\varphi)dxdy$ and integrate over the region bounded by the curve (C). Integration by parts yields

$$F(k)\int_{y_1}^{y_2} \Big/_{B}^{B'} (1-\tau)^{\beta} \frac{\partial\varphi}{\partial x}\,dy + F(k)\int_{x_1}^{x_2} \Big/_{D}^{D'} (1-\tau)^{\beta} \frac{\partial\varphi}{\partial y}\,dx$$

$$-\iint F'(\varphi)\,(1-\tau)^{\beta}\left\{\left(\frac{\partial\varphi}{\partial x}\right)^{2}+\left(\frac{\partial\varphi}{\partial y}\right)^{2}\right\}dxdy = 0 \qquad \text{(a)}$$

where the symbol $\Big/_{B}^{B'}$ indicates that the function under the integral is the difference of the values $(1-\tau)^{\beta}\frac{\partial\varphi}{\partial x}$ at the points B' and B (fig. 1) and similarly for $\Big/_{D}^{D'}$. Since, in the case of a maximum, φ increases in passing inward from the contour (C) at points B, B' the result is $\frac{\partial\varphi}{\partial x}dx > 0$ in passing within the region of integration. But since, at the first of these points $dx > 0$ and at the second < 0 for the motion along BB', then

$$\left(\frac{\partial\varphi}{\partial x}\right)_{B} > 0, \qquad \left(\frac{\partial\varphi}{\partial x}\right)_{B'} < 0 \quad \text{and} \quad \Big/_{B}^{B'} \frac{\partial\varphi}{\partial x} < 0$$

and similarly

$$\Big/_{D}^{D'} \frac{\partial\varphi}{\partial y} < 0$$

The function $F(\varphi)$ is chosen so that $F(\varphi)$ and $F'(\varphi)$ are, everywhere within (C), greater than zero. Turning now to the above-derived relation (a) it can be seen that all its terms are less than zero and therefore impossible, hence also the assumption of a maximum φ. (If the author assumed a minimum, the signs in the substitutions would have been reversed, and the function F chosen so that everywhere $F(\varphi)$ and $F'(\varphi) < 0$, and again would have arrived at the impossibility of (a).

The same consideration proves the correctness of the derived theorem also for the other above-mentioned conditions, formulas (8) being required for functions $x(\varphi, \psi)$ and $y(\varphi, \psi)$.

With the aid of equation (10) it is not difficult to prove a similar theorem also for the function τ and therefore the velocity of the flow likewise cannot have a maximum in the range of variables φ, ψ; a minimum may exist but the minimum value of τ is zero. In order to prove this the following equation is constructed on the basis of formulas (10)

$$\frac{\partial}{\partial\varphi}\frac{1-(2\beta+1)\tau}{\tau}(1-\tau)^{-\beta-1}\frac{\partial\tau}{\partial\varphi}+\frac{\partial}{\partial\psi}\frac{(1-\tau)^{\beta}}{\tau}\frac{\partial\tau}{\partial\psi}=0 \qquad \text{(b)}$$

Assume that there exists in the φ, ψ plane a point where τ has a maximum or minimum. Take, about this point, a curve (C) with constant value of τ; multiply the equation for τ by a certain function $f(\tau)$ and integrate the left part over the area bounded by the curve (C). Integrating by parts yields the relation:

$$\int\!\!/\;\frac{1-(2\beta+1)\tau}{\tau}(1-\tau)^{-\beta-1}f(\tau)\frac{\partial\tau}{\partial\varphi}\,d\psi+\int\!\!/\;\frac{(1-\tau)^{\beta}}{\tau}f(\tau)\frac{\partial\tau}{\partial\psi}d\varphi$$

$$-\iint\left\{\frac{1-(2\beta+1)\tau}{\tau}(1-\tau)^{-\beta-1}\left(\frac{\partial\tau}{\partial\varphi}\right)^{2}+\frac{(1-\tau)^{\beta}}{\tau}\left(\frac{\partial\tau}{\partial\varphi}\right)^{2}\right\}f'(\tau)\,d\varphi\,d\psi=0$$

In quite the same manner, as in the above-considered cases, the impossibility of this relation will be proved. It is readily seen, however, that the proof will be valid only for the condition: $\tau \leqq \frac{1}{2\beta+1}$ within the region of flow, and this condition has already been assumed and its physical meaning explained. The case $\tau = 0$ is itself excluded from the range of applicability of the above considerations and for the following reason: At the point $\tau = 0$, if this point lies within the flow mass, the streamlines meet. It is readily seen that in this case the coordinates x, y cannot be single-valued functions of φ, ψ; the latter region will be represented, at least, by a two-sheet Riemann surface not assumed in setting up the double integrals that figure in these considerations. It is easy to show, however, without any formulas, that the value $\tau = 0$ is the minimum τ. For this, it is sufficient to remember that $\tau = (u^2 + v^2)/2\alpha$, and, since this function is everywhere positive, the value zero is actually its minimum. In what follows, only such gas flows for which the critical point $\tau = 0$ lies on the bounding contour of the τ, θ region and the surface of the φ,ψ region, a single sheet surface will be considered.

By setting up formulas (10) the differential equation:

$$\frac{\partial}{\partial\varphi}\,\tau(1-\tau)^{-\beta}\,\frac{\partial\theta}{\partial\varphi} + \frac{\partial}{\partial\psi}\,\frac{\tau(1-\tau)^{\beta+1}}{1-(2\beta+1)\tau}\,\frac{\partial\theta}{\partial\psi} = 0$$

and applying the above-described device it is found that <u>the function $\theta(\varphi,\psi)$ cannot have either a maximum or a minimum.</u> In the same way the absence of turning values also for the functions φ, ψ, of τ and θ, if the latter are taken as the independent variables is established. For this purpose formulas (11) must be used.

From the foregoing theorems proved it is clear that in the φ, ψ region there cannot exist closed curves along which the functions x, y, τ, θ maintain constant values; all such curves must end at the boundary of the region. Similiar considerations hold for the τ, θ region and the curves φ = constant and ψ = constant.

In application only such problems as correspond to a τ, θ region bounded by concentric circles and the straight line segments passing through their centers will be kept in mind. The magnidutes τ, θ will be taken as the polar coördinates of the points of their region and the common center of the boundary curves will be the pole of the coordinates.

For these conditions from the theorem on the impossibility of a maximum or minimum of $\tau(\varphi,\psi)$ $\theta(\varphi,\psi)$ it may be concluded that inner points of the τ, θ region correspond to inner points of the φ, ψ region. It may be noted further that by making use of the absence of a maximum or a minimum of the function $\psi(\tau, \theta)$, there can again be obtained the theorem already proved on the uniqueness of the fuuction if it is continuous within the τ, θ region and is given on its boundaries. <u>The series of its boundary values may, in general, also be discontinuous</u>.

A problem on the flow of a gas will now be considered. Assume as known the corresponding contour of the region of the variables τ, θ satisfying the condition $\tau \leq 1/(2\beta + 1)$; finally ψ on the contour is known. If it is possible, from certain considerations, to conclude that the given problem has a solution and if a continuous function ψ satisfying the given conditions is found, then this function will actually represent the stream function, since no other is possible — because of the theorem on the uniqueness of the solution of the differential equations of the same type as the equations for the function ψ. (See also reference 6.)

Side considerations, as to the existence of a solution, are not, however, always a priori possible, and such being the case, having obtained a function ψ and through it φ, it may be questioned as to whether these particular functions give a possible solution of the problem. In order to remove such doubt it is necessary to show each time that the formulas for φ and ψ determine τ and θ as single-value functions of x and y.

In order to clarify this point, the reasoning will be as follows. Let a single-valued function $\psi(\tau, \theta)$ be defined; then from the formula

$$\frac{\partial x}{\partial \tau} = \frac{\partial x}{\partial \varphi}\frac{\partial \varphi}{\partial \tau} + \frac{\partial x}{\partial \psi}\frac{\partial \psi}{\partial \tau}$$

and similarly on the basis of relation (8) and formulas

$$\frac{\partial x}{\partial \varphi} = \frac{\cos\theta}{\sqrt{2\alpha\tau}}, \quad \frac{\partial y}{\partial \varphi} = \frac{\sin\theta}{\sqrt{2\alpha\tau}}$$

there is obtained

$$\sqrt{2\alpha\tau}\,\frac{\partial x}{\partial \tau} = \frac{\partial \varphi}{\partial \tau}\cos\theta - \frac{\partial \psi}{\partial \tau}\sin\theta\,(1-\tau)^{-\beta}$$

and similar ones.

Thus the derivatives of $x(\tau, \theta)$ and $y(\tau,\theta)$ are determined as single-valued functions of τ and θ. <u>If the Jacobian $(x, y)/(\tau, \theta)$ is not zero within the region of τ, θ, these, as is known, are defined as single-valued functions of x,y.</u> But

$$\left(\frac{x,\ y}{\tau,\ \theta}\right) = \left(\frac{x,y}{\varphi,\psi}\right)\left(\frac{\varphi,\psi}{\tau,\theta}\right)$$

and from equations (8) and (11) the relations

$$\left(\frac{x,y}{\varphi,\psi}\right) = \frac{(1-\tau)^{-\beta}}{\tau},$$

$$-\left(\frac{\varphi,\psi}{\tau,\theta}\right) = 2\tau(1-\tau)^{-\beta}\left(\frac{\partial\psi}{\partial\tau}\right)^2 + \frac{1-(2\beta+1)\tau}{2\tau}(1-\tau)^{-\beta-1}\left(\frac{\partial\psi}{\partial\theta}\right)^2$$

from which it is clear that, if everywhere $\tau \lesseqgtr 1/(2\beta+1)$, the equation $(x,y)/(\tau,\theta) = 0$ is possible only if both partial derivatives of the function ψ are simultaneously zero. This can happen at a singular point of one of the curves ψ = constant if such singular point on the curve exists. In general, to deny the existence of such points is impossible, but it can be stated that curves $\psi(\tau, \theta)$ = constant will in no case form a loop, since closed curves ψ = constant would then exist within the loop. Hence, the branches of our curve, after forming the singular point, will support themselves against the boundary of the region somewhat as shown in figure 2. If, however, it is known, at least from the conditions of the problem, that all the curves ψ = constant issue from the same point of the contour τ, θ and again meet at another point of the boundary, then the above-mentioned disposition of the curve is impossible and therefore the vanishing of the $(x,y)/(\tau,\theta)$ is likewise impossible. The same is also true in the case where, starting from the same point of the boundary of the region, the curves $\psi(\tau, \theta)$ = constant then divide into pencils, each of them again converging at one point.

An entirely different picture will result if a steady gas flow under conditions so that τ exceeds the limiting value $1/(2\beta+1)$ is sought. The Jacobian $(\varphi, \psi)/(\tau, \theta)$ in the region of τ, θ where τ is greater than the limiting value will then be the difference between two positive quantities and will become zero along a certain curve. Consider, for example, the case where to the boundaries of the φ, ψ region there corresponds in the τ, θ region the semicircle ACB and its diameter AB, the center of the semicircle being at $\tau = 0$; let ψ along this contour have some constant value.

Along OA evidently $(\partial\psi/\tau, \theta) = 0$; on the semicircle ACB $(\partial\psi/\partial\theta) = 0$. Therefore, in passing along any curve from a point M on the diameter to a point N on the semicircle, the ratio $\left(\frac{\partial\psi}{\partial\tau}\right)^2 : \left(\frac{\partial\psi}{\partial\theta}\right)^2$ passes through all possible values from 0 to ∞; hence it follows that if some value $\tau_0 \geq 1/(2\beta+1)$ is chosen for τ, then on each of the curves joining M and N a point will be found at which the expression

$$2\tau_0 \left(\frac{\partial\psi}{\partial\tau}\right)^2 + \frac{1-(2\beta+1)\tau_0}{2\tau_0} (1-\tau_0) \left(\frac{\partial\psi}{\partial\theta}\right)^2$$

becomes zero. The series of these points in the τ, θ region will be on a certain curve. The point where the latter meets the curve $\tau = \tau_0$ will be the point at which there holds the equation

$$(1-\tau)^\beta \left(\frac{\psi, \varphi}{\tau, \theta}\right) = 2\tau \left(\frac{\partial\psi}{\partial\tau}\right)^2 + \frac{1-(2\beta+1)\tau}{2\tau} (1-\tau) \left(\frac{\partial\psi}{\partial\theta}\right)^2 = 0$$

and therefore by the preceding formulas also the equation

$$\left(\frac{x, y}{\tau, \theta}\right) = 0.$$

Evidently from the manner in which one of these points is obtained it must be concluded that they form a certain dense curve. Thus $\tau(x,y)$ and $\theta(x,y)$ will not be single-valued as is required in a real motion of the gas. <u>Thus, if from the conditions of the problem, it is possible to conclude that the pressure of the gas flow and the velocity of its particles exceed the limit defined by the inequality $\tau < 1/(2\beta+1)$, then steady motion is, at least, not always possible.</u>

The author returns to the solution of the problem which was especially thought of in setting up this analysis. The flow of a gas bounded by plane walls at which the gas separates and continues to flow in a region of constant pressure is considered. The problems of the flow of a gas out of a very large vessel and the pressure of an infinite gas flow at a plate will be studied in greater detail.

Consider a particular solution of equation (12) of the form

$$\psi_n = z_n \sin(2n\theta + \alpha_n) \tag{14}$$

where z_n is a function only of τ. To determine this function, the ordinary differential equation

$$\frac{d}{d\tau}\left\{\tau(1-\tau)^{-\beta}\frac{dz_n}{d\tau}\right\} - \frac{1-(2\beta+1)\tau}{\tau(1-\tau)}(1-\tau)^{-\beta} n^2 z_n = 0 \tag{15}$$

is used, or, explicitly

$$\tau^2(1-\tau)\frac{d^2z_n}{d\tau} + \tau[1+(\beta-1)\tau]\frac{dz_n}{d\tau} - n^2[1-(2\beta+1)\tau]z_n = 0 \tag{16}$$

Setting

$$z_n = \tau^n y_n, \quad \text{if} \quad n > 0 \tag{17}$$

yields, for the determination of y_n, the equation

$$\tau(1-\tau)\frac{d^2y_n}{d\tau^2}+[2n+1+(\beta-2n-1)\tau]\frac{dy_n}{d\tau}+\beta n(2n+1)y_n=0 \quad (18)$$

This is a hypergeometric equation. Its integrals are of the form

$$y_n^{(1)} = K(\tau), \qquad y_n^{(2)} = \tau^{-2n} \; K(\tau)$$

where $K(\tau)$ denotes the series

$$c_0 + c_1\tau + c_2\tau^2 + \ldots$$

If it is desired to have an expression for ψ that does not become infinite at the critical point $\tau = 0$, in equation (14), it is necessary to take the integral of equation (16), which remains finite for $\tau = 0$. It, therefore is assumed that, by making use of the notation of Gauss

$$y_n = F(a_n, b_n, 2n + 1, \tau) \quad (19)$$

where a_n and b_n are determined from the equations

$$a_n + b_n = 2n-\beta, \quad a_n b_n = -\beta n(2n+1)$$

The question to be decided is which of the problems of the above-indicated type may be solved with the aid of a function ψ expressed by the formula

$$\psi = A + B\theta + \Sigma B_n\psi_n \quad (20)$$

where A, B, B_n are certain constants and ψ_n is determined by formulas (14), (17), and (19).

First consider the boundary conditions of the problems. Since the gas mass is to be bounded by stream-

$$\tau(1-\tau)\frac{d^2y_n}{d\tau^2} + [2n+1+(\beta-2n-1)\tau]\frac{dy_n}{d\tau} + \beta n(2n+1)y_n = 0 \quad (18)$$

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where A, B, B_n are certain constants and ψ_n is determined by formulas (14), (17), and (19).

First consider the boundary conditions of the problems. Since the gas mass is to be bounded by streamlines, then along the bounding contour of the τ,θ region ψ must assume certain constant values. If the part of the contour under consideration corresponds to a plane wall, the angle θ formed by the velocity direction with the x-axis should maintain a constant value so that this part of the boundary will be a section of a straight line passing through the pole $\tau = 0$. If the surface of the jet is considered, there is along the surface p = constant, and therefore, by the Bernoulli theorem, the velocity should likewise have a constant value v_0. But $v^2/2\alpha = \tau$, so that τ likewise has a constant value τ_0. It is clear that the part of the boundary of the τ, θ region corresponding to the jet will be formed of an arc of a circle the center of which serves as the pole.

The problem proposed of the motion of a gas mass is now compared with the corresponding problem of the flow of an incompressible liquid for the same boundary conditions (the same disposition of the boundary walls, velocities at infinity, and velocity at the jet boundaries). The latter problem is solved with the aid of the well-known Joukowsky method. By the use of this method the relation between the complex variables $\lg v_0/v + i\theta = \lg \sqrt{\tau_0/\tau} + i\theta$ and $w = \varphi_1 + i\psi_1$ is found where φ_1 and ψ_1 are the velocity potential and the stream function corresponding to the problem. It is assumed that

$$w = f\left(\lg \sqrt{\frac{\tau_0}{\tau}} + i\theta\right) \tag{21}$$

is obtained and that this function can be expanded in a series of the form

$$w = k + B\left(\lg \sqrt{\frac{\tau_0}{\tau_0}} + i\theta\right) + \Sigma k_n \left(\frac{\tau}{\tau_0}\right)^n e^{2ni\theta}$$

Then

$$\psi_1 = A + B\theta + \Sigma B_n \left(\frac{\tau}{\tau_0}\right)^n \sin\,(2n\theta + \alpha_n) \tag{22}$$

It is asserted that the corresponding problem in the case of the gas flow is solved by the formula

$$\lambda\psi = A + B\theta + \Sigma B_n \left(\frac{\tau}{\tau_o}\right)^n \frac{y_n}{y_{n,o}} \sin(2n\theta + \alpha_n) \quad (23)$$

where y_n is the hypergeometric series defined by formula (19), $y_{n,o}$ its particular value with τ substituted for τ_o and λ a certain constant.

The correctness of this statement may, in part, be proved immediately. Thus, it is readily seen that for $\tau = \tau_o$ the right-hand sides of formulas (22) and (23) agree; hence if for $\tau = \tau_o$ ψ_1 = constant, then likewise ψ = constant. If, further, for any value $\theta = \theta_o$ the function defined by formula (22) does not depend on τ, this is true only if the condition $\sin(2n\theta_o + \alpha_n) = 0$ is satisfied for every n under the summation sign; but then the right side of formula (23) for the same θ also will have a constant value. Thus the boundary conditions imposed on the function ψ are satisfied.

It is now noted here the the series ψ formally satisfies equation (12), since it is the sum of its partial integrals. If now it is shown that for any $\tau < \tau_o$ the series (23) converges and for $\tau = \tau_o$ tends to the same limit as series (22), then the function expressed by it actually will be the required stream function. If, moreover, it is shown that this series converges absolutely and uuiformly together with the series obtained by its term-by-term differentiation with respect to τ and θ it will be justifiable to consider the latter series as expressions for the partial derivatives of the initial series. Then for a given ψ, making use of equations (7'), (8') and (11), φ, x, y, will be found. As regards φ, it is found from formulas (11) which lead to the relation

$$d\varphi = - B \frac{1-(2\beta+1)\tau}{2\tau} (1-\tau)^{-\beta-1} d\tau$$

$$+ \sum \frac{B_n}{\tau_o^n y_{n,o}} \left[2\sin(2n\theta+\alpha_n) d\theta z'_n \tau (1-\tau)^{-\beta} \right.$$

$$\left. -2n(\cos 2n\theta + \alpha_n) d\tau z_n \frac{1-(2\beta+1)\tau}{2\tau} (1-\tau)^{-\beta-1} \right]$$

whence by use of equation (15) and the following ones equation (24)

$$\varphi = C + B(1-\tau)^{-\beta} - \frac{B}{2}\int \frac{(1-\tau)^{-\beta}}{\tau}\,d\tau$$

$$- (1-\tau)^{-\beta}\Sigma B_n \left(\frac{\tau}{\tau_o}\right)^n \frac{y_n}{y_{n,o}} \left(1 + \frac{\tau}{n}\frac{y'_n}{y_n}\right) \cos(2n\theta + \alpha_n) \quad (24)$$

may be easily obtained.

The functions $1 + \frac{\tau}{n}\frac{y'_n}{y_n}$ which, in what follows, will be denoted by x_n, play a very important part in this problem since through them are expressed the characteristic constants of the various problems. Certain properties of these functions and the methods for their computation and likewise the essential properties of the functions z_n and y_n of interest here will be described in the following section. Only by becoming acquainted with all these properties is the possibility obtained of demonstrating the correctness of these statments that remain to be proved with regard to the fundamental series for ψ.

PART II

CERTAIN PROPERTIES OF THE FUNCTIONS z_n, y_n, AND x_n.

PROOF OF CONVERGENCE OF THE SERIES FOR ψ AND φ.

The function z_n is that integral of the equation

$$\tau\frac{d}{d\tau}\tau(1-\tau)^{-\beta}\frac{dz_n}{d\tau} = n^2[1-(2\beta+1)\tau](1-\tau)^{-\beta-1}z_n \quad (25)$$

which does not become infinite for $\tau = 0$. This integral is of the form $z_n = \tau^n y_n$ where $n > 0$ and y_n is the hypergeometric function

$$y_n = F(a_n,\ b_n,\ 2n+1,\ \tau)$$

the parameters a_n and b_n being defined by the formulas

$$a_n + b_n = 2n - \beta, \quad a_n b_n = -\beta n(2n+1)$$

It will be shown first of all that z_n <u>does not possess any real roots between the values 0 and $\frac{1}{2\beta+1}$ of the variable τ</u>. Assume the contrary to be the case and let $\tau = a$ be the least positive root of the function z_n. Since z_n becomes zero also for $\tau = 0$, then, between the values 0 and a, a quantity b should exist which serves as the root of the equation $\frac{dz_n}{d\tau} = 0$. Thus the function under the differentiation sign on the left-hand side of equation (25) will have the roots $\tau = 0$, $\tau = b$, and consequently its derivative should possess a root $\tau = c$ where $0 < c < b < a$. In view of the fact that on the right-hand side of this equation the coefficient of z_n cannot become zero for $\tau < \frac{1}{2\beta+1}$, it must be assumed that $z_n(c) = 0$; and hence the function z_n must have a root $\tau = c < a$. By the same reasoning it is concluded that in the range of values of the variables from 0 to a an infinite number of roots of z_n must be included. But then z_n could not be expressed by a power series.

From the proposition just proved, it follows that z_n is an increasing function. Thus, since z'_n has no roots less than $\frac{1}{2\beta + 1}$, z_n must always vary in the same sense. Since it is positive for very small τ, the same sign will be maintained for all values of the variable within the above-mentioned range. Thus z_n increases for values of τ near zero, and hence will continue to increase until z'_n changes sign. It is noted that

$$z'_n = z_n\left(\frac{n}{\tau} + \frac{y'_n}{y_n}\right) = \frac{z_n n}{\tau} x_n$$

where it is concluded that, <u>for $0 < \tau < \frac{1}{2\beta + 1}$, the functions x_n have no roots and are always greater than 0</u> for within these limits $z'_n > 0$.

Turn now to the function y_n, the holomorphic integral of the equation

$$y''_n \tau(1 - \tau) + y'_n[(2n + 1)(1 - \tau) + \beta\tau] + \beta n(2n + 1)y_n = 0 \qquad (26)$$

or its equivalent

$$\frac{d}{d\tau}\tau^{2n+1}(1 - \tau)^{-\beta} y'_n + \beta n(2n + 1)\tau^{2n}(1 - \tau)^{-\beta-1} y_n = 0 \qquad (26')$$

From the theorem just proved, it is concluded that <u>y_n does not have any roots between the values of τ within the range considered.</u> The same may be proved likewise with regard to the successive derivatives of this function. First of all, from equation (26') it is concluded that what has been stated is true with regard to the function y'_n. Thus, if there existed a root of this function the derivative

$$\frac{d}{d\tau}\tau^{2n+1}(1-\tau)^{-\beta}y'_n$$

would likewise have a root. But this is impossible because the second term on the left-hand side of equation (26') cannot become zero within the range of variation of τ. By differentiating m times equation (26), there is obtained

$$y_n^{(m+2)}\tau(1-\tau) + y_n^{(m+1)}[(2n+m+1)(1-\tau) - (m-\beta)\tau]$$

$$+ [\beta n(2n+1) - m(2n-\beta) - m^2]y_n^{(m)} = 0$$

or

$$\frac{d}{d\tau}\tau^{2n+m+1}(1-\tau)^{m-\beta}y_n^{(m+1)} + [\beta n(2n+1) - m(2n-\beta)$$

$$- m^2]\tau^{2n+m}(1-\tau)^{m-\beta-1}y_n^{(m)} = 0$$

whence, reasoning as before, it is concluded that $y_n^{(m+1)}$ cannot have roots within the range of variation of τ if $y_n^{(m)}$ does not have roots within this range. By setting $m = 1, 2, 3 \ldots,$ the correctness of this statement is proved.

Setting $\tau = 0$, gives

$$y_n(0) = 1,\quad y'_n(0) = -\beta n,\quad y''_n(0) = \frac{\beta n}{2n+2}[\beta n(2n+1) - 2n + \beta - 1]$$

$$y'''_n(0) = -\frac{\beta n}{2n+2}\,\frac{\beta n(2n+1) - 2n + \beta - 1}{2n+3}[\beta n(2n+1) - 4n + 2\beta - 4]$$

Since β is approximately 2.5 and n is a positive number, the signs of the above quantities alternate. The same will be true for any τ satisfying the inequalities

$0 < \tau < \frac{1}{2\beta + 1}$. Thus $y_n > 0$, $y'_n < 0$, $y''_n > 0$, $y'''_n < 0$ and the quantities y_n, y'_n, y''_n numerically decrease.

Consider now the function $s_n = y_n(1 - \tau)^{-m}$. The differential equation which this equation satisfies can be easily derived from (26) and is of the form

$$(1 - \tau)\frac{d}{d\tau}\tau^{2n+1}(1 - \tau)^{2m-\beta} s'_n + \tau^{2n}(1 - \tau)^{2m-\beta}\Big[(\beta n - m)(2n + 1)$$

$$+ m(m - \beta - 1)\frac{\tau}{1 - \tau}\Big]s_n = 0 \qquad (27)$$

By setting $m = \beta n$ it is seen that for $n > 1 + \frac{1}{\beta}$ the quantity within the brackets maintains the plus sign whatever the value of τ*. If n is an integer this is true for all $n \gtreqless 2$; for $n = 1$ it will have the minus sign. It is assumed that $n > 1 + \frac{1}{\beta}$.

By setting $m = \mu\beta n$ for a suitable choice of μ, there is in the brackets a negative quantity for all values of τ if the expression is negative or zero for the largest admissible value of τ: namely, $\frac{1}{2\beta + 1}$. For this it is sufficient to choose μ so that it satisfies the equation

$$(1 - \mu)(2n + 1) + \frac{\mu}{2\beta}(\mu\beta n - \beta - 1) = 0$$

or, after reduction,

$$\mu^2 n - 2\mu\left(2n + \frac{3}{2} + \frac{1}{2\beta}\right) + 2(2n + 1) = 0 \qquad (28)$$

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*Within the limits, of course, of 0 and $\frac{1}{2\beta + 1}$; this must be kept in mind throughout the following discussion.

The maximum value of μ is obtained for very large n ($n = \infty$); in this case $\mu = 2$. In general $\mu < 2$; thus, for $n = 2$, $\beta = 2.5$, $\mu = 1.083$; for $n = 3$, $\mu = 1.181$.

With the above choice of the quantity m the function s'_n cannot have a root different from zero within the range of variation of τ. This can be proved by the same reasoning already more than once applied.

It is noted that $s_n(0) = 1$ and from equation (27) $s'_n(0) = m - \beta n$ is obtained. For $m = \mu\beta n$, $s'_n(0) > 0$ and therefore the function

$$y_n(1 - \tau)^{-\mu\beta n} \tag{29}$$

increases with the variable and will exceed unity. If $m = \beta n$, however, $s'_n(0) = 0$, but $s''_n(0) = -\beta n \dfrac{\beta n - \beta - 1}{2n + 2}$ and this magnitude for $n > 1 + \dfrac{1}{\beta}$ is a negative quantity. For this reason $s'_n(\tau)$ likewise, as a decreasing function, should for $\tau > 0$ be less than zero; hence it is concluded that

$$y_n(1 - \tau)^{-\beta n} \tag{30}$$

for $n > 1 + \dfrac{1}{\beta}$ is a decreasing function and represents a proper fraction.

For $n < 1 + \dfrac{1}{\beta}$ the quantity within the brackets in equation (27) will be less than 0 for $m = \beta n$; $s'_n(0) = 0$, $s''_n(0) > 0$, and therefore $s'_n(\tau) > 0$ and

$$y_n(1 - \tau)^{-\beta n} \tag{29'}$$

will be an increasing function.

The smaller root, denoted simply by μ, of equation (28) will be less than 1. It can be readily shown that if $m = \mu\beta m$ and $0 < \tau < \dfrac{1}{2\beta + 1}$, the coefficient of s_n

in equation (27) will again be of constant sign, the latter being positive. Therefore $s'_n(\tau) > 0$ is obtained since $s'_n(0) > 0$; and hence it is concluded that the function

$$y_n(1 - \tau)^{-\mu\beta n} \qquad (30')$$

decreases with increasing τ. Thus, for example, for $n = 1$, $\mu = 0.93$ (β is taken equal to 2.5), and therefore with increasing $y_1(1 - \tau)^{-\beta}$, $y_1(1 - \tau)^{-0.93\beta}$ will be a decreasing function.

The above-mentioned properties of y_n give limiting functions within which y_n is included: namely,

$$\text{for} \quad n > 1 + \frac{1}{\beta}, \quad (1 - \tau)^{\beta n} > y_n > (1 - \tau)^{\mu\beta n}$$

$$\text{for} \quad n < 1 + \frac{1}{\beta}, \quad (1 - \tau)^{\beta n} < y_n < (1 - \tau)^{\mu\beta n}$$

where μ is determined by equation (28) and is equal to the smaller of its roots.

It may be noted further that the function

$$y_n(1 - \tau)^{-2\beta n}$$

increases a fortiori. For $n < 1 + \frac{1}{\beta}$ this is evident; for $n > 1 + \frac{1}{\beta}$, on the basis of what has been said above, it may be considered as the product of two increasing functions $y_n(1 - \tau)^{-\mu\beta n}$ and $(1 - \tau)^{-(2-\mu)\beta n}$. Hence if the greatest value which τ assumes in the given problem is denoted by τ_o and the corresponding value of y_o by $y_{n,o}$, this will give the inequality

$$\frac{y_n(1 - \tau)^{-2\beta n}}{y_{n,o}(1 - \tau_o)^{-2\beta n}} < 1$$

or

$$\frac{y_n}{y_{n,0}} < \left(\frac{1 - \tau}{1 - \tau_0}\right)^{2\beta n} \qquad (31)$$

Last to be considered is the function $x_n = 1 + \frac{\tau}{n}\frac{y'_n}{y_n}$ on which, as has been shown in the foregoing, depends the computation of the very important constants of interest in the various problems. The differential equation which the function x_n satisfies is first set up. It is obtained from the hypergeometric equation (26) by setting

$$y_n = e^{\int_0^{\tau} \frac{x_n - 1}{\tau} n \, d\tau}$$

Thus, it is found that

$$x'_n\tau(1 - \tau) + nx_n^2(1 - \tau) + x_n\beta\tau - n[1 - (2\beta + 1)\tau] = 0 \qquad (32)$$

This equation, together with the condition $x_n(0) = 1$, fully determines the function x_n. It has been shown already that the function x_n for a change in the variable within the limits under consideration remains always greater than zero. It will be shown <u>that it decreases with increase in</u> τ. For this purpose equation (32) was differentiated; there was obtained

$$x''_n\tau(1 - \tau) + x'_n[2n(1 - \tau)x_n + (\beta - 1)\tau + 1 - \tau]$$

$$= nx_n^2 - x_n\beta - n(2\beta + 1)$$

Substituting in the brackets for x_n its value $1 + \frac{\tau}{n}\frac{y'_n}{y_n}$ and multiplying the equation by $\tau^{2n}(1 - \tau)^{-\beta}y_n^2$, reduces it to the form

$$\frac{d}{d\tau}\tau^{2n+1}(1 - \tau)^{-\beta+1}y_n^2x'_n = [nx_n^2 - x_n\beta - n(2\beta + 1)]\tau^{2n}(1 - \tau)^{-\beta}y_n^2$$

But x_n is less than 1, since $x_n - 1 = \frac{\tau}{n}\frac{y'_n}{y_n}$ is a negative magnitude on the basis of what has been said with regard to the signs of the function y_n and its derivatives. Hence, the right-hand side of the obtained equation is a negative quantity of constant sign. If x'_n, equal to $-\beta$ for $\tau = 0$, had a root within the range of variation of τ, then $\frac{d}{d\tau}\tau^{2n+1}(1-\tau)^{-\beta+1} y_n^2 x'_n$ would also become zero for a value of τ less than this root, a result which is impossible. But x'_n, everywhere finite, as can readily be shown, cannot change sign except by passing through a root. Thus x'_n remains less than zero and therefore x_n decreases.

The next step is to seek to obtain functions that <u>limit</u> the value of x_n. For this purpose the following theorem will be proved: <u>If, on substituting in the equation determining x_n, a holomorphic function k_n, there is obtained on the left side a positive value of constant sign, then $k_n > x_n$; the inequality sign will be reversed if the result of the substitution is less than zero.</u> For $\tau = 0$, k_n may be equal to 1. From the assumed inequality

$$k'_n\tau(1-\tau) + nk_n^2(1-\tau) + k_n\beta\tau - n[1-(2\beta+1)\tau] \gtrless 0 \qquad (33)$$

subtract equation (32), which leaves

$$(k'_n - x'_n)\tau(1-\tau) + (k_n - x_n)[\beta\tau + n(1-\tau)(k_n + x_n)] \gtrless 0 \qquad (33')$$

By setting

$$k_n = 1 + \frac{\tau}{n}\frac{l'_n}{l_n}; \qquad l_n = e^{\int_0^\tau \frac{k_n - 1}{\tau} n\,d\tau} > 0$$

then, on substituting in the brackets for k_n and x_n their values in terms of l_n and y_n and multiplying by the positive factor $y_n l_n \tau^{2n-1}(1-\tau)^{-\beta}$, there is obtained

$$\frac{d}{d\tau}(k_n - x_n)\tau^{2n}(1-\tau)^{-\beta} y_n l_n \gtrless 0$$

Integrating this inequality within the limits 0 to τ yields

$$(k_n - x_n)\tau^{2n} (1 - \tau)^{-\beta} y_n l_n \gtrless 0$$

whence the required inequality is obtained

$$k_n \gtrless x_n$$

It will now be shown how the function x_n may be computed to any degree of accuracy by transforming it into a continued fraction. In differential equation (32) the new independent variable s defined by $s = \frac{\tau}{1 - \tau}$ is substituted; when τ varies from 0 to $\frac{1}{2\beta + 1}$, s varies from 0 to $\frac{1}{2\beta}$. The differential equation for x_n' will become

$$x'_n s(1 + s) + x_n\beta s + nx_n^2 - n(1 - 2\beta s) = 0 \qquad (34)$$

From equation (34) is found $x_n(0) = 1$, $x'_n(0) = -\beta$. If any function k_n satisfies inequality (33), then on substituting in (34) there will be obtained

$$k'_n s(1 + s) + k_n\beta s + nk_n^2 - n(1 - 2\beta s) \gtrless 0 \qquad (35)$$

whence follows as before the relation $k_n \gtrless x_n$, for $k_n(0) = 1$.

Equation (34) together with inequality (35) will be written as

$$x'_n s(1 + s) + x_n\beta s + nx_n^2 - n(1 - 2\beta s) \substack{= \\ >} 0 \qquad (36)$$

which is to be understood as follows: If, on substituting any finite function within the range of variation of s and equal to 1 for $s = 0$, the result is zero on the left-hand side of relation (36), then this function is the exact expression for x_n; if, as a result of the substitution a positive quantity is obtained, the substituted function is always greater than x_n; in the contrary case the sign of the inequality is reversed.

$$x_n = 1 - \frac{\beta s}{1 - c^{(n)} s} \tag{37}$$

where $c^{(n)}$ is a new function to be determined. On reducing and changing sign, there results from (36)

$$c^{(n)'} s(1 + s) + c^{(n)} (2n + 2 - \beta s) - (2n + 1) c^{(n)^2} s - n\beta + \beta + 1 \gtreqless 0 \tag{37'}$$

where it was necessary to reverse the inequality sign. The meaning of the relation is as follows: If, after substituting in the left-hand side any function in place of $c^{(n)}$, the result is a negative quantity; then replacing $c^{(n)}$ in formula (37) by this value there is obtained an upper limit of the function x_n - that is, a function greater than x_n.

Further is set

$$c^{(n)} = \frac{c_o^{(n)}}{1 - \partial^{(n)} s} \tag{38}$$

where $c_o^{(n)} = c^{(n)}(0) = \frac{\beta}{2} - \frac{2\beta + 1}{2n + 2}$, and the function ∂ satisfies the relation

$$\partial^{(n)'} s(1 + s) + \partial^{(n)}[2n + 3 + (\beta + 1)s] - \partial^{(n)^2} s(2n + 2)$$

$$- (2n + 1) c_o^{(n)} - \beta \gtreqless 0 \tag{38'}$$

whence is obtained

$$\partial_o^{(n)} = \partial^{(n)}(0) = \frac{\beta}{2} + \frac{2\beta + 1}{2n + 2} - 2\frac{2\beta + 1}{2n + 3}$$

Next, setting successively

$$\partial^{(n)}=\frac{\partial_0^{(n)}}{1-c_1^{(n)}s},\quad c_1^{(n)}=\frac{c_{1,0}^{(n)}}{1-\partial_1^{(n)}s},\quad \partial_1^{(n)}=\frac{\partial_{1,0}^{(n)}}{1-c_2^{(n)}s},$$

(39)

$$c_2^{(n)}=\frac{c_{2,0}^{(n)}}{1-\partial_2^{(n)}s},\quad \partial_2^{(n)}=\frac{\partial_{2,0}^{(n)}}{1-c_3^{(n)}s}\ldots\ldots,$$

yields, for the determination of c_1, ∂_1, c_2, d_2..., the relations:

$$c_1^{(n)\prime}s(1+s)+c_1^{(n)}(2n+4-\beta s)-(2n+3)c_1^{(n)2}s-$$
$$-\partial_0^{(n)}(2n+2)+\beta+1\gtreqless 0.$$

$$\partial_1^{(n)\prime}s(1+s)+\partial_1^{(n)}[2n+5+(\beta+1)s]-(2n+4)\partial_1^{(n)2}s-$$
$$-c_{1,0}^{(n)}(2n+3)-\beta\gtreqless 0,$$

$$c_2^{(n)\prime}s(1+s)+c_2^{(n)}(2n+6-\beta s)-(2n+5)c_2^{(n)2}s-$$
$$-\partial_{1,0}^{(n)}(2n+4)+\beta+1\gtreqless 0,$$

$$\partial_2^{(n)\prime}s(1+s)+\partial_2^{(n)}[2n+7+(\beta+1)s]-(2n+6)\partial_2^{(n)2}s-$$
$$-c_{2,0}^{(n)}(2n+5)-\beta\gtreqless 0.$$

Moreover

$$c^{(n)}_{1,0}=\frac{\beta}{2}+2\frac{2\beta+1}{2n+3}-4\frac{2\beta+1}{2n+4},\quad \partial^{(n)}_{1,0}=\frac{\beta}{2}+4\frac{2\beta+1}{2n+4}-6\frac{2\beta+1}{2n+5},$$

$$c^{(n)}_{2,0}=\frac{\beta}{2}+6\frac{2\beta+1}{2n+5}-9\frac{2\beta+1}{2n+6},\quad \partial^{(n)}_{2,0}=\frac{\beta}{2}+9\frac{2\beta+1}{2n+6}-12\frac{2\beta+1}{2n+7},$$

With the aid of these formulas it is not difficult to set up equations for the determination of $c^{(n)}_m$, $\partial^{(n)}_m$, and formulas for $c^{(n)}_{m,o}$, $\partial^{(n)}_{m,o}$ The latter are of the form

$$c^{(n)}_{m,0}=\frac{\beta}{2}+m(m+1)\frac{2\beta+1}{2n+2m+1}-(m+1)^2\frac{2\beta+1}{2n+2m+2},$$

$$\partial^{(n)}_{m,0}=\frac{\beta}{2}+(m+1)^2\frac{2\beta+1}{2n+2m+2}-(m+1)(m+2)\frac{2\beta+1}{2n+2m+3},$$

or, after reduction:

$$c^{(n)}_{m,0}=\frac{\beta}{2}-(2\beta+1)\frac{(m+1)(2n+m+1)}{(2n+2m+1)(2n+2m+2)},$$

(40)

$$\partial^{(n)}_{m,0}=\frac{\beta}{2}-(2\beta+1)\frac{(m+1)(2n+m+1)}{(2n+2m+2)(2n+2m+3)}.$$

The equations for determining c_m and ∂_m are the following:

$$c^{(n)\prime}_m s(1+s)+c^{(n)}_m(2n+2m+2-\beta s)-(2n+2m+1)c^{(n)2}_m s-$$
$$-\partial^{(n)}_{m-1,0}(2n+2m)+\beta+1\gtreqless 0,$$

$$\partial^{(n)\prime}_m s(1+s)+\partial^{(n)}_m[2n+2m+3+(\beta+1)s]-(2n+2m+2)\partial^{(n)2}_m s-$$
$$-c^{(n)}_{m,0}(2n+2m+1)-\beta\gtreqless 0.$$

All these relations can be readily verified by the method of passing from m to m + 1.

The sign $<$ in the last of the foregoing relations holds until the index m exceeds a certain limiting value: namely, while $c^{(n)}_{m,o}$ is positive; if $c^{(n)}_{m,o} < 0$ then in the relation for $\partial^{(n)}_{m}$ the sign $<$ must be replaced by $>$. This is because, among the simplifications which were made in transforming the above relation, there occurred division by $c^{(n)}_{m,o}$.

By collecting the results, x_n is finally expressed by the formula:

$$x_n = 1 - \cfrac{\beta s}{1 - \cfrac{c^{(n)}_{o} s}{1 - \cfrac{\partial^{(n)}_{o} s}{1 - \cfrac{c^{(n)}_{1,o} s}{1 - \partial^{(n)}_{1,o} s, \ldots}}}} \tag{41}$$

where $c^{(n)}_{m,o}$ and $\partial^{(n)}_{m,o}$ are expressed by formulas (40).

Now consider the magnitude of the quantities $c^{(n)}_{m,o}$, $\partial^{(n)}_{m,o}$. It is not difficult to see that they are always contained between $\beta/2$ and $-\lambda$. The first of these is obtained for n very large ($n = \infty$); the second, in general, differs little from $-1/4$ and is obtained from the minimum of the expression

$$c^{(n)}_{k-1,o} = \frac{\beta}{2} - \frac{(2\beta + 1)k(2n + k)}{2(2n + 2k - 1)(n + k)}$$

as a function of k. This minimum occurs either for $k = En\sqrt{4n^2 - 1} + n(2n - 1)$, or $k = En\sqrt{4n^2 - 1} + n(2n - 1) + 1$; these values for integral n are equal to

$$4n^2 - n - 1 \quad \text{and} \quad 4n^2 - n$$

The coefficients of $\frac{2\beta + 1}{2}$ in the formula for $c^{(n)}_{k-1,o}$ correspondingly receive the values

$$\frac{1}{2 - \dfrac{2n^2 - 1}{(4n^2 - n - 1)(4n^2 - 1 + n)}} \quad \text{and} \quad \frac{1}{2\left(1 - \dfrac{1}{16n^2 - 1}\right)}$$

the second of which is larger than the first. Thus the mininum value of $c^{(n)}{}_{k-1.0}$ is

$$\frac{\beta}{2} \frac{2\beta + 1}{4\left(1 - \dfrac{1}{16n^2 - 1}\right)}$$

It can further easily be shown that the continued fraction (41) is always convergent. The contrary could be the case only if the expression

$$\cfrac{c^{(n)}{}_{o}s}{1 - \cfrac{\partial^{(n)}{}_{o}s}{1 - c^{(n)}{}_{1.0}s \ldots}}$$

approached unity. But even in the least favorable case, for $n = \infty$, this quantity becomes

$$\cfrac{\frac{1}{2}\beta s}{1 - \cfrac{\frac{1}{2}\beta s}{1 - \frac{1}{2}\beta s \ldots}}$$

and its maximum value, obtained for the maximum value $s = \frac{1}{2\beta}$ is

$$\cfrac{\frac{1}{4}}{1 - \cfrac{\frac{1}{4}}{1 - \frac{1}{4} \ldots}} = \frac{1}{2}$$

As regards the signs of c and ∂ (for simplicity in writing, the indices are omitted) those for which m is equal to 0, 1, 2 ... up to a certain limiting value will be positive, all the remaining ones negative, provided that n has a finite value. The limiting value of $1 + m$ is obtained from the inequality

$$\frac{\beta}{2} - (2\beta + 1)\frac{k(2n + k)}{(2n + 2k - 1)(2n + 2k)} < 0$$

or, on reducing,

$$k^2 + (2n + \beta)k - \beta n(2n - 1) > 0$$

This inequality is satisfied as soon as $m + 1 = k$ exceeds the larger of the roots of the equation

$$\sigma^2 + (2n + \beta)\sigma - \beta n(2n - 1) = 0$$

The limiting value of m will therefore be expressed by the formula

$$m = E\left\{-n - \frac{\beta}{2} + \sqrt{n^2\ (2\beta + 1) + \frac{\beta^2}{4}}\right\}$$

If only integral values of n are considered, the following limiting m and $c^{(n)}{}_{m,o}$ will be obtained:

$$n = 1, \quad m = 0, \quad c^{(1)}{}_{o} = -\frac{1}{4}$$

$$n = 2, \quad m = 1, \quad c^{(2)}{}_{1.o} = -\frac{1}{28}$$

$$n = 3, \quad m = 3, \quad c^{(3)}{}_{3.o} = -\frac{17}{4\times 7\times 13} \qquad \beta \text{ assumed equal to } 2.5$$

$$n = 4, \quad m = 4, \quad c^{(3)}{}_{4.o} = -\frac{5}{3\times 4\times 7} \tag{42}$$

m increasing with n as required. The quantity $\partial^{(n)}{}_{m,o}$ likewise becomes negative, but only for large values of m. For $\partial^{(n)}{}_{m,o}$ to be less than zero, the inequality

$$\frac{\beta}{2} - (2\beta + 1)\frac{k(2n + k)}{(2n + 2k + 1)(2n + 2k)} < 0$$

must be satisfied, or

$$k^2 + (2n - \beta)k - \beta n(2n + 1) > 0$$

The limiting value of m denoted by m_1 is expressed by the formula

$$m_1 = E\left\{-n + \frac{\beta}{2} + \sqrt{n^2(2\beta + 1) + \frac{\beta^2}{4}}\right\}$$

But if $p > q$ and $p = E(p) + \theta$, $q = E(q) + \vartheta$ where θ and ϑ are proper fractions, then $p - q = E(p) - E(q) + \theta - \vartheta$; thus

$$E(p - q) = E(p) - E(q), \quad \text{or} \quad E(p) - E(q) - 1$$

Therefore, comparing the obtained values of m_1 and the limiting m, it will be found that m_1 will be equal to the limiting m for the coefficient $c^{(n)}{}_{m,o}$ plus $E(\beta)$, or plus $E(\beta) + 1$. Thus

$$\left.\begin{aligned} n &= 1, & m_1 &= 2, & \partial^{(1)}{}_{2,o} &= 0 \\ n &= 2, & m_1 &= 4, & \partial^{(2)}{}_{4,o} &= -\frac{1}{28} \\ n &= 3, & m_1 &= 5, & \partial^{(3)}{}_{5,o} &= -\frac{1}{76} \\ n &= 4, & m_1 &= 7, & \partial^{(4)}{}_{7,o} &= -\frac{3}{100} \end{aligned}\right\} \beta = 2.5 \qquad (43)$$

All $c^{(n)}{}_{m,o}$ starting from that which corresponds to the limiting value of m like all $\partial^{(n)}{}_{m,o}$ for $m \gtreqless m_1$ are negative quantities and the remainder $1 - \dfrac{\partial^{(n)}{}_{m_1,o}s}{1 - c^{(n)}{}_{m_1+1,o}s}$ of the continued fraction is expressed in the usual form

$$1 + \cfrac{a}{1 + \cfrac{b}{1 + c \ldots}}$$; its numerical value being contained between $1 + a$ and $1 + \frac{a}{1 + b}$.

Of the functions x_n the one that is particularly simple is x_1, which for the assumed value of β is expressed as a fraction of two polynomials of the third degree. For $n = 1$

$$c^{(1)}{}_{0} = -\frac{1}{4}, \quad \partial^{(1)}{}_{0} = \frac{7}{20}, \quad c^{(1)}{}_{1,0} = -\frac{7}{20}, \quad \partial^{(1)}{}_{1,0} = \frac{3}{28}$$

$$c^{(1)}{}_{2,0} = -\frac{5}{14}, \quad \partial^{(2)}{}_{2,0} = 0 \quad (\text{by } (43))$$

$$x_1 = 1 - \cfrac{5s/2}{1 + \cfrac{s/4}{1 - \cfrac{7s/10}{1 + \cfrac{7s/20}{1 - \cfrac{3s/28}{1 + 5s/14}}}}}$$

or, after reducing,

$$x_1 = \frac{32 - 64s - 14s^2 - 2s^3}{(4 + s)(s^2 + 2s + 8)} \tag{44}$$

With this formula y_1 is readily found. For this purpose the previous variable $\tau = \frac{s}{1 + s}$ is substituted which gives

$$x_1 = 1 + \tau \frac{y'_1}{y_1} = \frac{32 - 160\tau + 210\tau^2 - 84\tau^3}{32 - 80\tau + 70\tau^2 - 21\tau^3} \tag{44'}$$

whence, since $y_1(0) = 1$, there is obtained

$$32y_1 = (4 - 3\tau)(8 - 14\tau + 7\tau^2)$$

The preceding simplification results only from the rounded values: $\beta = 2.5$, $\gamma = 1.40$. The more accurate value of γ for air is 1.4025 and β is somewhat less than 2.5. In this case all x_n are expressed by infinite continued fractions.

To employ in the applications the exact formulas for x_n appears impossible, since this would offer very great difficulties which have not been overcome. However, by using even the simplest proper fractions, x_n is obtained with sufficient accuracy. Entirely satisfactory results are obtained even in the case limited to the third proper fraction and x_n is expressed by

$$x_n = 1 - \cfrac{\beta s}{1 - \cfrac{c^{(n)}{}_0 s}{1 - \partial^{(n)}{}_0 s}} \tag{45'}$$

or, after reduction and substitution of the values of the coefficients

$$x_n = 1 - \beta s - \frac{\beta s^2 (2n + 3)(\beta n - \beta - 1)}{(2n + 2)[2n + 3 - (2\beta n - \beta - 2)s]} \tag{45}$$

The error for such computation of x_n is greater the greater the value of s. The magnitude of this error now is estimated, <u>considering only the integral values of</u> n.

With the exact value of x_n for $n = 1$, a direct comparison of the results of the computation of this function may be carried out by formulas (44) and (45). The computation will be made for maximum $s = \frac{1}{2\beta}$ or $s = 0.2$, assuming as before $\beta = 2.5$. The exact value of x_1 will be

$$\frac{776}{1477} = 0.5253893$$

From formula (45) is obtained

$$x_1 = 0.525510$$

The difference is approximately 0.00012.

For other values of n in estimating the error, it is necessary to proceed otherwise. It is noted, first of all, that for all $n > 2$ formula (45') gives a function greater than x_n; the contrary is true for $n = 2$. In order to show this, turn to equation (38'), determining $\partial^{(n)}$. This equation may be written as follows:

$$\partial^{(n)'} s(1 + s) + \partial^{(n)}[2n + 3 + (\beta + 1) s] - \partial^{(n)^2} s(2n + 2)$$
$$- (2n + 3)\partial^{(n)}{}_o = 0$$

Substituting in the left-hand side $\partial^{(n)}{}_o$ for $\partial^{(n)}$, gives

$$\partial^{(n)}{}_o s[\beta + 1 - (2n + 2)\partial^{(n)}{}_o]$$

or, after substituting in the brackets for $\partial^{(n)}{}_o$ its value by formulas (40)

$$\partial^{(n)}{}_o s\left[1 - \beta n + \frac{(2\beta + 1)(2n + 1)}{2n + 3}\right] = \partial^{(n)}{}_o s\left[2 - \beta(n-2) - \frac{2(2\beta + 1)}{2n + 3}\right]$$

For $n > 2$ this value will be less than 0. But from this, as has been said, it must be concluded that on replacing $\partial^{(n)}$ in the formula for x_n by a trial value, a function greater x_n is obtained. On the contrary, for $n = 2$ in the brackets, the quantity

$$2 - 2\frac{2\beta + 1}{7} > 0$$

which shows the correctness of the reversed inequality (x_2 is greater than the value that would be obtained on substituting $\partial^{(2)}{}_o$ for ∂^2 in the formula for x_2) for the value $n = 2$.

As regards x_2, the lower limit of the function will be the following proper fraction. Thus

$$1 - \beta s - \frac{\beta s^2 c^{(2)}_0}{1 - (c^{(2)}_0 + \partial^{(2)}_0)s} < x_2 < 1 - \beta s$$

$$- \frac{\beta s^2 c^{(2)}_0}{1 - (c^{(2)}_0 + \partial^{(2)}_0)s - \dfrac{d^{(2)}_0 c^{(2)}_{1,0} s^2}{1 - c^{(2)}_{1,0} s}}$$

By computing the values of the limiting functions for the case of the greatest difference in their values, when $s = \frac{1}{2\beta} = 0.2$, there is obtained

$$0.47034 < x_2(0.2) < 0.47037$$

In the case $n > 2$, assume in the equation for $\partial^{(n)}$, $\partial^{(n)} = \frac{\partial^{(n)}_0}{1 - ks}$, where k is a constant to be determined, and k is chosen so that the result of the substitution is greater than zero for $0 < s < \frac{1}{2\beta}$. This requirement leads to the inequality

$$k(2n + 4 - \beta s) - k^2 s(2n + 3) - (2n + 4)c^{(n)}_{1,0} \geqq 0$$

The smaller root on the left-hand side of this inequality is expressed by

$$\frac{n + 2 - \dfrac{\beta s}{2} - \sqrt{(n + 2)(1 - 2\beta s) \pm 4s(n + 1)(2\beta + 1) + \dfrac{\beta^2 s^2}{4}}}{(2n + 3)s}$$

Its maximum value corresponds to $s = \frac{1}{2\beta} = 0.2$ and is equal to

$$5\,\frac{n + 1.75 - \sqrt{\dfrac{24(n + 1)}{5} + \dfrac{1}{16}}}{2n + 3} \qquad (46)$$

If k is equal to this value, the above inequality will be satisfied.

It is now possible to indicate the limits within which x_n is included. The upper limit is expressed by formula (45'); the lower is obtained by substituting in this formula $\frac{\partial(n)_o}{1 - ks}$ for $\partial^{(n)}{}_o$. After reducing, finally

$$1 - \beta s - \frac{\beta s^2(\beta n - \beta - 1)}{(2n + 2)\left(1 - \frac{2\beta n - \beta - 2}{2n + 3}s\right)} > x_n > 1 - \beta s$$

$$- \frac{\beta s^2(\beta n - \beta - 1)}{(2n + 2)\left(1 - \frac{2\beta n - \beta - 2}{2n + 3}s - \lambda s^2\right)} \tag{47}$$

where $\lambda = \frac{k\partial_o}{1 - ks}$, $\partial_o = \frac{\beta}{2} - \frac{(2\beta + 1)(2n + 1)}{(2n + 2)(2n + 3)}$, and k is determined by (46).

The numerical values are given for $s = 0.2$ of the limiting functions for x_n for $n = 3, 4, 5$, and 6:

$$0.4348 > x_3(0.2) > 0.4343$$

$$0.4095 > x_4(0.2) > 0,4073$$

$$0.3905 > x_5(0.2) > 0.3872$$

$$0.3755 > x_6(0.2) > 0.3704$$

It is thus seen that the error increases, or more accurately, may increase with n but nevertheless is very small for small values of the latter. For somewhat large values of n the limits of error widen. Thus

$$n = 12, \quad 0.326 > x_{12}(0.2) > 0.308$$

$$n = 24, \quad 0.293 > x_{24}(0.2) > 0.251$$

This unfavorable circumstance is offset, however, to some extent by the fact that the functions x_n with large n enter the more removed terms of the series and the coefficients of these terms are relatively small.

The limiting functions for x_n also will be given with large n. These functions will be useful in computing the limits within which the remainder term of the series for the gas jet problem is included. Again, in the differential equation defining x_n

$$x'_n s(1 + s) + x_n \beta s + n x^2_n - n(1 - 2\beta s) = 0$$

Substitute on the left-hand side the expression $\sqrt{1 - 2\beta s + 2us^2}$, and choose the function u so that the result of the substitution is greater than zero. Then, by the theorem proved above,

$$\sqrt{1 - 2\beta s + 2us^2} > x_n$$

This substitution gives on the left-hand side of the equation for x_n the expression

$$k = \frac{u's(1 + s) + 2u[1 + (1 + \beta)s] - \beta(1 + 2\beta) + 2nu\sqrt{1 - 2\beta s + 2us^2}}{\sqrt{1 - 2\beta s + 2us^2}} s^2 \quad (47)$$

which, as can be seen, will be greater than zero if

$$u = \beta\sqrt[3]{\frac{\beta(1 + 2\beta)^2}{2n^2}} = \frac{5}{2}\sqrt[3]{\frac{45}{n^2}}$$

Thus, for this value of u, $\sqrt{1 - 2\beta s + 2us^2}$ is a decreasing function of s; the product, however, of this root by $2nu$ for the maximum value of the variable $s = \frac{1}{2\beta}$ is equal to $(1 + 2\beta)\beta$, and therefore the quantity k remains positive. On the other hand, it becomes negative, whatever the value of s, if

$$u = \frac{\beta(1 + 2\beta)}{2n + 2}$$

since the numerator in the expression for k in this case is equal to zero for $s = 0$ and, as a decreasing function, will be less than zero for $s > 0$. Thus

$$\sqrt{1 - 2\beta s + 2\beta s^2 \sqrt[3]{\frac{\beta(1 + 2\beta)^2}{2n^2}}} > x_n > \sqrt{1 - 2\beta s + \frac{\beta s^2(1 + 2\beta)}{n + 1}} \qquad (48)$$

By raising somewhat the upper limit of the function x_n, the first part of the double inequality also can be transformed into

$$\sqrt{1 - 2\beta s} + \sqrt[3]{\frac{2\beta^2(1 + 2\beta)}{n}}s > x_n$$

and therefore x_n can be expressed by the formula

$$x_n = \sqrt{1 - 2\beta s} + \lambda s \sqrt[3]{\frac{2\beta^2(1 + 2\beta)}{n}} \qquad (49)$$

where λ is a proper fraction.

It can be easily shown that x_n for the same value of the variable decreases with increasing n. This is clear from the equation for x_{n+m}:

$$x'_{n+m}s(1 + s) + x_{n+m}\beta s + (n + m)x^2_{n+m} - (n + m)(1 - 2\beta s) = 0$$

Substituting on the left x_n for x_{n+m}, there is obtained on the basis of the equation for x_n

$$m[x^2_n - (1 - 2\beta s)]$$

a magnitude greater than zero due to inequality (48), and therefore it is concluded that

$$x_n > x_{n+m} \qquad (50)$$

whatever the positive number m.

Now with the properties of the function x_n that are of importance for what follows, two inequalities which the functions y_n must satisfy will be noted further. The first of these will be derived in the following manner: Set up the differential equation determining $\eta_n = \frac{y'_n}{y_n}$:

$$(\eta'_n + \eta^2_n)\tau(1 - \tau) + [(2n + 1)(1 - \tau) + \beta\tau]\eta_n + \beta n(2n + 1) = 0$$

Also set $\eta_n = \xi_n - \frac{2\beta n}{1 - \tau}$, which gives

$$(\xi'_n + \xi^2_n)\tau(1 - \tau) + [(2n + 1)(1 - \tau) - \beta(4n - 1)\tau]\xi_n$$

$$- \beta n(2n + 1) + \frac{2\beta n\tau}{1 - \tau}(2\beta n - \beta - 1) = 0 \qquad (51)$$

To this equation, as can be easily seen, the theorem proved for equation (36) for the function x_n is applicable. If on substituting for ξ_n any holomorphic function there is obtained on the left-hand side an expression greater than zero, the substituted expression will be greater than ξ_n. If $2\beta n$ is substituted this result in fact is obtained, and therefore $\xi_n < 2\beta n$.

Substitute, further, in the equation defining ξ_{n+m} in place of ξ_{n+m} the function ξ_n. The result of the substitution, on the basis of equation (51), which is satisfied by ξ_n, reduces to

$$m\frac{1 - (2\beta + 1)\tau}{1 - \tau}[2\xi_n(1 - \tau) - \beta(4n + 1) - 2\beta m] - \frac{(2\beta + 1)2\beta m\tau}{1 - \tau}$$

This expression is negative for any τ, since it gives a result less than zero on substituting for ξ_n the greater magnitude $2\beta n$. Hence $\xi_n < \xi_{n+m}$, or

$$\frac{y'_n}{y_n} + \frac{2\beta n}{1 - \tau} < \frac{y'_{n+m}}{y_{n+m}} + \frac{2\beta(n + m)}{1 - \tau}$$

Integrating this inequality from τ to τ_0, and passing from logarithms to numbers, gives

$$\frac{y_n}{y_{n,o}}\left(\frac{1-\tau}{1-\tau_o}\right)^{-2\beta n} > \frac{y_{n+m}}{y_{n+m,o}}\left(\frac{1-\tau}{1-\tau_o}\right)^{-2\beta(n+m)} \tag{52}$$

where $y_{n,o}$, by the assumed notation, denotes $y_n(\tau_o)$. Thus the function $\frac{y_n}{y_{n,o}}\left(\frac{1-\tau}{1-\tau_o}\right)^{-2\beta n}$ decreases with increasing n.

The second inequality which it was proposed to derive follows from relation (50). From the latter is obtained

$$\frac{y'_n}{y_n} > \frac{y'_{n+m}}{y_{n+m}}$$

Integrating within the limits s and s_o, gives the result on passing from logarithms to numbers:

$$\frac{y_n}{y_{n,o}} < \frac{y_{n+m}}{y_{n+m,o}} \tag{53}$$

Thus, the ratio $\frac{y_n}{y_{n,o}}$ increases with increasing n.

It is necessary to proceed to the proof of the convergence of the series giving the solution of the gas jet problems. In explaining the general method of solution of this type of problem (see pt. I), the following formulas for expressing the stream function and the velocity potential were arrived at:

$$\psi = B\theta + \sum B_n\left(\frac{\tau}{\tau_o}\right)^u \frac{y_n}{y_{n,o}} \sin(2n\theta + \alpha_n)$$

$$\varphi = B(1-\tau)^{-\beta} - \frac{B}{2}\int (1-\tau)^{-\beta}\frac{d\tau}{\tau}$$

$$-(1-\tau)^{-\beta}\sum B_n\left(\frac{\tau}{\tau_o}\right)^u \frac{y_n}{y_{n,o}} x_n \cos(2n\theta + \alpha_n)$$

These are formulas (23) and (24) of part I. The indices n entering them increase as the terms of an arithmetic progression. It will be shown the preceding series are absolutely and uniformly convergent for any $\tau < \tau_0$ if this is true of the series

$$\psi_1 = B\theta + \sum B_n \left(\frac{\tau}{\tau_0}\right)^n \sin (2n\theta + \alpha_n)$$

$$\varphi_1 = B - \frac{B}{2}\int\frac{d\tau}{\tau} - \sum B_n \left(\frac{\tau}{\tau_0}\right)^n \cos (2n\theta + \alpha_n)$$

expressing the stream function and velocity potential for the corresponding problem in the case of incompressible liquids. The series ψ_1 and φ_1 will evidently be absolutely convergent if $\lim_{n = \infty} \frac{B_{n_1}}{B_n} < 1$, where n and n_1 are two successive values of n. It will be assumed that this condition is satisfied. On the basis of relation (31) it can then be stated that the terms of the series ψ are correspondingly less than the terms of the series

$$\sum [B_n] \left\{\frac{\tau(1 - \tau)^{2\beta}}{\tau_0(1 - \tau_0)^{2\beta}}\right\}^n \tag{54}$$

which is an absolutely convergent series for $\tau < \tau_0$ if $\tau_0 < \frac{1}{2\beta + 1}$; for in that case

$$\frac{d}{d\tau}\tau(1 - \tau)^{2\beta} = [1 - (2\beta + 1)\tau](1 - \tau)^{2\beta-1} > 0$$

and therefore $\frac{\tau(1 - \tau)^{2\beta}}{\tau_0(1 - \tau_0)^{2\beta}}$ is a proper fraction.

The remainder term of the series ψ

$$R_n = \sum_{n=n'}^{n=\infty} B_n \left(\frac{\tau}{\tau_0}\right)^n \frac{y_n}{y_{n,0}} \sin (2n\theta + \alpha_n)$$

is numerically less than the term R'_n of the series (54).

$$R'_n = \sum_{n=n}^{n=\infty}[B_n]\left\{\frac{\tau(1-\tau)^{2\beta}}{\tau_o(1-\tau_o)^{2\beta}}\right\}^n$$

but R'_n approaches zero with increasing n no matter in what manner τ is less than τ_o. From this it is concluded that the series ψ is uniformly convergent.

Since the series entering the function φ differs from the one just considered by having cosines instead of sines and the successive terms multiplied by a series of decreasing positive quantities, the theorems just proved likewise hold for the series φ. Furthermore, it can be easily seen that the same properties are possessed by the derivatives of the functions φ and ψ with respect to θ and therefore also their derivatives with respect to τ, since the latter are connected with the former by the linear relations (11) of part I. A consequence of these theorems is the continuity of the functions φ and ψ and their derivatives within the range of the variables τ, θ under consideration. (See vol I, p. 310 of reference 7.)

It will be shown, finally, that as τ approaches the limiting value τ_o, the series φ and ψ approach limits which are the values of the series obtained on substituting τ_o for τ. For this purpose, consider the sums σ and σ' of p terms after the nth in the series φ and ψ. Let $n_1, n_2 \ldots n_p$ be successive values of n; then denoting the fractions

$$\frac{\tau}{\tau_o}\left(\frac{1-\tau}{1-\tau_o}\right)^{2\beta}, \quad \frac{y_n}{y_{n,o}}\left(\frac{1-\tau}{1-\tau_o}\right)^{-2\beta n}$$

correspondingly by ξ and η_n, these sums can be transformed into

$$\sigma = -(1-\tau)^{-\beta}\sum_{n_1}^{n_p} B_n\,\xi^n\eta_n x_n \cos\,(2n\theta + \alpha_n)$$

$$\sigma' = \sum_{n_1}^{n_p} B_n\,\xi^n\,\eta_n \sin\,(2n\theta + \alpha_n)$$

The expressions

$$B_n \xi^n \cos (2n\theta + \alpha_n), \quad B_n \xi^n \sin (2n\theta + \alpha_n)$$

will now be denoted by u_n and u'_n.

The series Σu_n and $\Sigma u'_n$ uniformly converge for any ξ less than unity, since they coincide with the series entering the functions φ_1 and ψ_1. It is assumed that they converge also for $\xi = 1$; then from a known theorem in analysis their value for $\xi = 1$ is the limit which they approach as ξ approaches 1 (and hence $\tau \rightarrow \tau_0$). But in this case n can be given an increasingly large value so that the sums

$$u_{n_1}, u_{n_1} + u_{n_2}, u_{n_1} + u_{n_2} + u_{n_3}, \ldots u_{n_1} + u_{n_2} + \ldots + u_{n_p}$$

are included between any values ϵ and δ as small as is desired, whether the quantity ξ is less than or equal to unity. And, since on the basis of the properties investigated in this section of the functions x_n and y_n, the quantities η_n, x_n entering the expressions σ and σ' are greater than zero and decrease with increasing n, then by the theorem of Abel, σ is included between the limits

$$-(1-\tau)^{-\beta} \delta\eta_{n_1} \quad \text{and} \quad -(1-\tau)^{-\beta} \epsilon\eta_{n_1}$$

For the same reasons σ' is included between other arbitrarily small numerical limits, and the proposition is thus proved.

As a result of all the properties which have been demonstrated of the series φ and ψ the conclusion is arrived at, which was the object of the investigations: namely, that <u>the formulas obtained are an actual solution of the proposed gas flow problems</u>.

PART III

THE FLOW OF A GAS FROM AN INFINITELY WIDE VESSEL

The method described will be applied to the problem of the flow of a gas from an infinite vessel with plane walls, the simplest case being considered – that is, where one wall is a continuation of the other.

Consider an incompressible liquid flowing out of such a vessel (fig. 4); AB and A'B' are the traces of the walls of the vessel; OX is the trace of its plane of symmetry; BCC'B' is the escaping jet. If the quantity flowing out per second is denoted by Q, the velocity potential and the stream function, respectively, by φ_1 and ψ_1, considering $\psi = 0$ on OX, then in the region of flow φ varies from $-\infty$ to $+\infty$ and ψ from $-\frac{Q}{2}$ to $+\frac{Q}{2}$. The complex variable

$$w = \varphi_1 + i\psi_1$$

will be connected with another complex variable u (reference 2) through the relation

$$w = \frac{Q}{\pi} \lg \frac{u}{ci}$$

The region of variation of w then corresponds to the upper half plane of the region of u. In addition, the logarithm of the ratio of velocities v_0/v at the jet surface and at the point of the fluid considered will be denoted by ϑ, and the angle of the velocity direction with the X axis by θ. The problem is then solved if

$$\vartheta + i\theta = i \arcsin \frac{c}{u}$$

For, on the boundary ABC $\psi = -\frac{Q}{2}$, φ varies from $-\infty$ to $+\infty$ and u passes through the negative part of the real axis from 0 to $-\infty$; $\theta = \frac{\pi}{2}$, $\infty > \vartheta > 0$ if

$0 > u > -c$; $\vartheta = 0$, $v = v_0$, $\frac{\pi}{2} > \theta > 0$ for $-c > u > -\infty$. On the boundary A'B'C' $\psi = \frac{Q}{2}$, φ varies within the same limits; $\theta = -\frac{\pi}{2}$, $\infty > \vartheta > 0$ for $0 < u < c$; $\vartheta = 0$, $v = v_0$, $-\frac{\pi}{2} < \theta < 0$ if $c < u < \infty$. The point $u = 0$ thus corresponds to the infinitely distant point of the vessel where $\vartheta = \infty$ and the velocity becomes zero; $u = \infty$ gives the part of the jet at infinity. Finally, for $\theta = 0$, u is purely imaginary and $\psi = 0$ and the center line of flow coinciding with the X axis is obtained.

From the preceding formulas is found

$$w = -\frac{Q}{\pi} \lg \left(i \sin \frac{\vartheta + i\theta}{i}\right)$$

or, if $\vartheta + i\theta$ is denoted by σ,

$$w = -\frac{Q}{\pi} \lg i \sin \frac{\sigma}{i} = -\frac{Q}{\pi} \lg \frac{e^{\sigma} - e^{-\sigma}}{2} = -\frac{Q}{\pi} (\sigma - \lg 2)$$

$$-\frac{Q}{\pi} \lg \left(1 - e^{-2\sigma}\right)^{*} \tag{55}$$

It is noted that $\varphi_1 = 0$ at the points where the jet separates from the walls; σ in these cases has the value $\pm i\frac{\pi}{2}$.

By expanding the logarithm in formula (55) in a series, there is obtained:

$$\pi \frac{\varphi_1 + i\psi_1}{Q} = \lg 2 - (\vartheta + i\theta) + \sum_{n=1}^{\infty} \frac{e^{-2n\vartheta}}{n}(\cos 2n\theta - i \sin 2n\theta)$$

whence

*In the case of a vessel with the walls meeting at an angle it is necessary to replace in this formula σ by $\frac{\sigma}{q}$ and in the succeeding relations ϑ by $\frac{\vartheta}{q}$, θ by $\frac{\theta}{q}$. The angle between the walls in this case will be equal to $q\pi$.

$$\frac{\pi}{Q}\psi_1 = -\theta - \sum_1^\infty \frac{e^{-2n\vartheta}}{n} \sin 2n\theta$$

Substituting in this formula the variable τ determined by the equation $\frac{\tau}{\tau_o} = \frac{v^2}{v^2_o} = e^{-2\vartheta}$ results in the required expression:

$$\frac{\pi}{Q}\psi_1 = -\theta - \sum_1^\infty \frac{1}{n}\left(\frac{\tau}{\tau_o}\right)^n \sin 2n\theta$$

Since this series is absolutely convergent, therefore, by the method given above, by use of formula (23) an expression is arrived at for the stream function ψ defining the flow of gas from a vessel of this kind; there is obtained:

$$\frac{\pi}{Q}\psi = -\theta - \sum_1^\infty \frac{1}{n}\left(\frac{\tau}{\tau_o}\right)^n \frac{y_n}{y_{n,o}} \sin 2n\theta \tag{56}$$

All curves ψ = constant in the τ,θ region start from the point $\tau = 0$ and meet again at the point $\tau = \tau_o$, $\theta = 0$.

The velocity potential by formula (24) is determined by the relation:

$$\frac{\pi}{Q}\varphi = C + \frac{1}{2}\int \frac{d\tau}{\tau(1-\tau)^\beta} - (1-\tau)^{-\beta}$$
$$+ (1-\tau)^{-\beta}\sum_1^\infty \frac{1}{n}\left(\frac{\tau}{\tau_o}\right)^n \frac{y_n}{y_{n,o}} x_n \cos 2n\theta$$

With the formula for φ, it is not difficult to set up the equation of the jet. By formulas (8) of part I

$$\frac{\cos\theta}{\sqrt{2\tau\alpha}}=\frac{\partial x}{\partial\varphi}=(1-\tau)^{\beta}\frac{\partial y}{\partial\psi},\quad \frac{\sin\theta}{\sqrt{2\tau\alpha}}=\frac{\partial y}{\partial\varphi};$$

whence

$$\sqrt{2\tau\alpha}\frac{\partial y}{\partial\theta}=\sin\theta\frac{\partial\varphi}{\partial\theta}+(1-\tau)^{-\beta}\cos\theta\frac{\partial\psi}{\partial\theta}.$$

If use is made of the formulas for φ and ψ, there is obtained

$$\frac{\pi}{Q}\sqrt{2\tau\alpha}\frac{\partial y}{\partial\theta}=-2(1-\tau)^{-\beta}\sum_{1}^{\infty}\left(\frac{\tau}{\tau_0}\right)^n\frac{y_n}{y_{n,0}}(\cos 2n\theta\cos\theta+x_n\sin 2n\theta\sin\theta)-\cos\theta(1-\tau)^{-\beta}.$$

Integrating with respect to θ yields

$$\frac{\pi}{Q}\sqrt{2\tau\alpha}y=\Phi(\tau)-(1-\tau)^{-\beta}\sum_{1}^{\infty}\left(\frac{\tau}{\tau_0}\right)^n\frac{y_n}{y_{n,0}}\left[\frac{\sin(2n+1)\theta}{2n+1}+\frac{\sin(2n-1)\theta}{2n-1}\right]+$$

$$+(1-\tau)^{-\beta}\sum_{1}^{\infty}\left(\frac{\tau}{\tau_0}\right)^n\frac{y_n}{y_{n,0}}x_n\left[\frac{\sin(2n+1)\theta}{2n+1}-\frac{\sin(2n-1)\theta}{2n-1}\right]-\sin\theta(1-\tau)^{-\beta}.$$

Since, for $\theta = 0$, it should follow that $y = 0$; therefore $\Phi(\tau) = 0$. Thus, finally,

$$\frac{\pi}{Q}\sqrt{2\tau\alpha}\,y(1-\tau)^{\beta}=-\sin\theta+$$

$$+\sum_{1}^{\infty}\left(\frac{\tau}{\tau_0}\right)^n\frac{y_n}{y_{n,0}}x_n\left[\frac{\sin(2n+1)\theta}{2n+1}-\frac{\sin(2n-1)\theta}{2n-1}\right]-$$

$$-\sum_{1}^{\infty}\left(\frac{\tau}{\tau_0}\right)^n\frac{y_n}{y_{n,0}}\left[\frac{\sin(2n-1)\theta}{2n-1}+\frac{\sin(2n+1)\theta}{2n+1}\right] \qquad (57)$$

The series

$$\sum_{1}^{\infty}\left(\frac{\tau}{\tau_0}\right)^n\frac{y_n}{y_{n,0}}\left[\frac{\sin(2n-1)\theta}{2n-1}+\frac{\sin(2n+1)\theta}{2n+1}\right]$$

may be put into the form

$$\sum\xi^n\zeta_n\left[\frac{\sin(2n-1)\theta}{2n-1}+\frac{\sin(2n+1)\theta}{2n+1}\right], \qquad (58)$$

where

$$\xi=\frac{\tau(1-\tau)^{2\beta}}{\tau_0(1-\tau_0)^{2\beta}}<1,\quad \zeta_n=\frac{y_n(1-\tau)^{-2\beta n}}{y_{n,0}(1-\tau_0)^{-2\beta n}}.$$

By the theorem proved at the end of part II ζ_n represents a series of magnitudes decreasing with n. Thus the expression

$$\sum\xi^n\left[\frac{\sin(2n-1)\theta}{2n-1}+\frac{\sin(2n+1)\theta}{2n+1}\right]$$

and ξ approaches 1, or, in other words, as τ approaches τ_0, tends to the value

$$I=\sum\left[\frac{\sin(2n-1)\theta}{2n-1}+\frac{\sin(2n+1)\theta}{2n+1}\right],$$

and the same limit is approached by the more complicated summation (58) entering the formula for the coordinate y. Transforming I into

$$I=-\sin\theta+2\sum_{1}^{\infty}\frac{\sin(2n-1)\theta}{2n-1},$$

yields

$$2\sum_{1}^{\infty}\frac{\sin(2n-1)\theta}{2n-1}=R\lim_{t=0}2i\sum_{1}^{\infty}\frac{e^{-(2n-1)(t+i\theta)}}{2n-1}=$$

$$=R\lim_{t=0}i\lg\frac{1+e^{-t-i\theta}}{1-e^{-t-i\theta}}=R\left(i\lg\frac{1+\cos\theta-i\sin\theta}{1-\cos\theta+i\sin\theta}\right);$$

where R indicates that the real part of the expression must be taken. It can be readily seen that, for a continuous change of θ from 0 to θ,

$$\lg\frac{1+\cos\theta-i\sin\theta}{1-\cos\theta+i\sin\theta}=\lg\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}e^{-i\frac{\pi}{2}},$$

and therefore

$$2\sum_{1}^{\infty}\frac{\sin(2n-1)\theta}{2n-1}=\pm\frac{\pi}{2},$$

depending on the sign of θ; thus

$$I=\pm\frac{\pi}{2}-\sin\theta.$$

In view of the importance of the relation obtained, more rigorous method of its deviation will be presented. Starting from the equation

$$\sum_{1}^{m}\cos(2n-1)\theta=\frac{\sin 2m\theta}{2\sin\theta}$$

and integrating it within the limits 0 to θ results in

$$\sum_{1}^{m}\frac{\sin(2n-1)\theta}{2n-1}=\int_{0}^{\theta}\frac{\sin 2m\theta}{\theta}\frac{\theta}{\sin\theta}d\theta.$$

$$\lim_{m=\infty}\sum_{1}^{m}\frac{\sin(2n-1)\theta}{2n-1}=\frac{1}{2}\lim_{\mu=\infty}\int_{0}^{\theta}\frac{\sin\mu\theta}{\theta}\frac{\theta}{\sin\theta}d\theta.$$

But this limit, as is known, equals $\frac{\pi}{2}$ if θ is positive and $-\frac{\pi}{2}$ if it is negative. (See reference 7, vol. II, p. 233.) Hence,

$$I=-\sin\theta+2\sum_{1}^{\infty}\frac{\sin(2n-1)\theta}{2n-1}=-\sin\theta+2\lim_{m=\infty}\sum_{1}^{m}\frac{\sin(2n-1)\theta}{2n-1}=\pm\frac{\pi}{2}-\sin\theta.$$

The second series in equation (57), for $\tau = \tau_0$ approaches the expression

$$\sum_1^\infty x_n \left[\frac{\sin(2n-1)\theta}{2n-1} - \frac{\sin(2n+1)\theta}{2n+1}\right]$$

which is a convergent series for all values of θ. This can be shown by considering the remainder term of this series:

$$R_n = \sum_n^\infty x_n \left[\frac{\sin(2n-1)\theta}{2n-1} - \frac{\sin(2n+1)\theta}{2n+1}\right]$$

Substituting in the preceding equation for x_n its expression given by equation (49) yields

$$R_n = \sqrt{1-2\beta s}\,\frac{\sin(2n-1)\theta}{2n-1} + k\sum \frac{\lambda_n}{\sqrt[3]{n}}\left[\frac{\sin(2n-1)\theta}{2n-1} - \frac{\sin(2n+1)\theta}{2n+1}\right]$$

where k is a finite number and λ_n a proper fraction. Hence $\lim_{n=\infty} R_n = 0$.

By now putting $\tau = \tau_0$ in the formula for the coordinate y, the equation of the jet boundary is arrived at

$$\sqrt{2\alpha\tau_0}(1-\tau_0)^\beta \frac{\pi}{Q} y = \mp \frac{\pi}{2} + \sum_1^\infty x_{n,o}\left[\frac{\sin(2n+1)\theta}{2n+1} - \frac{\sin(2n-1)\theta}{2n-1}\right]$$

where the upper sign of $\frac{\pi}{2}$ corresponds to θ greater than zero.

If the width of the infinitely distant part of the jet is denoted by $2b$, then

$$2b\sqrt{2\alpha\tau_0}(1-\tau_0)^\beta = Q$$

for $\sqrt{2\alpha\tau_0}$ is the velocity at infinity, $\rho = \rho_0(1-\tau)^\beta$

$$\frac{\pi y}{2b} = \mp \frac{\pi}{2} - \sum_1^\infty \left[\frac{\sin(2n-1)\theta}{2n-1} - \frac{\sin(2n+1)\theta}{2n+1}\right] x_n \tag{59}$$

In order to determine the jet contraction, it must be noted that, for the conditions under consideration $\left(\tau_0 \lesseqgtr \frac{1}{2\beta+1}\right)$, the contraction occurs at infinity as in the case of the outflow of an incompressible liquid. Thus, if the maximum contraction were at a finite distance from the orifice, it would then be followed by an expansion: The streamlines would be turned by the concavity toward the inside of the jet; the pressure would drop from the surface inward and would reach a minimum at a certain point on the line of symmetry. At this point the velocity would receive its maximum value, which result is impossible. Thus the contraction will be equal to the ratio of the width $2b$ at infinity to the width $2a$ of the orifice of the vessel. This ratio is determined from formula (59) by substituting $\theta = \frac{\pi}{2}$, $y = -a$, and taking the upper sign of the first term on the right-hand side. Then there is obtained

$$\frac{\pi a}{2b} = \frac{\pi}{2} + \sum_{1}^{\infty} (-1)^{n-1} \frac{4n}{4n^2 - 1} x_{n,o}$$

whence the contraction is

$$\frac{b}{a} = \frac{\pi}{\pi + 8\sum_{1}^{\infty} (-1)^{n-1} \frac{n}{4n^2 - 1} x_{n,o}} \tag{60}$$

The series $S = \sum_{1}^{\infty} (-1)^{n-1} \frac{n}{4n^2 - 1} x_{n,o}$ must be convergent since it is an alternating series with numerically decreasing terms. Another way of proving it is by substituting for $x_{n,o}$ its expression, formula (49), of the preceding section. Thus the remainder term S of the series is found in the form

$$R_n = \sum_{n}^{\infty} (-1)^{n-1} \frac{n}{4n^2 - 1} \sqrt{1 - 2\beta s_o} + ks_o \sum_{n}^{\infty} (-1)^{n-1} \frac{\lambda_n}{n^{4/3}}$$

$$= \frac{1}{4}\sqrt{1 - 2\beta s_o}\, \frac{(-1)^{n-1}}{2n-1} + ks_o \sum_{n}^{\infty} (-1)^{n-1} \frac{\lambda_n}{n^{4/3}}$$

where it is clear that it approaches zero with increasing n, for k is a certain constant and λ_n a proper fraction.

For $\tau_0 = 0$, $x_{n0} = 1$ and this leads to Kirchhoff's formula

$$\frac{b}{a} = \frac{\pi}{\pi + 2} = 0.61$$

This will be the approximate value of the contraction for small flow velocities. To compute $\sum_1^\infty (-1)^{n-1} \frac{n}{4n^2 - 1} x_n$ for finite velocities and finite difference in pressure between the reservoir and the medium into which the discharge occurs, use is made of the approximate formula for x_n.

The limits between which the quantity $\frac{b}{a}$ is contained are found by taking an <u>even</u> or <u>odd</u> number of initial terms of the series

$$\sum_1^\infty (-1)^{n-1} \frac{n}{4n^2 - 1} x_n$$

In the first case the positive terms are replaced by the lower limiting values, and the negative terms by the upper limiting values; in the second case the reverse is done. This computation will be made for the limiting case $\tau_0 = \frac{1}{2\beta + 1}$ considering only five terms of the series for determining the upper limit and six terms for determining the lower. Use is made of the values of $x_1, \ldots x_6$ computed in part II, and the upper and lower limits of $\sum (-1)^{n-1} \frac{n}{4n^2 - 1} x_n$ are denoted by A and B, respectively, to find

$$B = \frac{0.5254}{3} - \frac{2 \times 0.4703}{15} + \frac{3 \times 0.4343}{35}$$

$$- \frac{4 \times 0.4095}{63} + \frac{5 \times 0.3872}{99} - \frac{6 \times 0.3755}{143}$$

whence, with an accuracy of 0.001,

$$B = 0.128$$

By rejecting the last term and adding the possible errors, A is obtained. The errors will add up only to 0.0003 and will have no effect on the accuracy desired. Hence

$$A = 0.128 + 0.015$$

Substituting these limiting values of the summation in formula (60) yields

$$\frac{\pi}{\pi + 1.14} < \frac{b}{a} < \frac{\pi}{\pi + 1.02}$$

or, if the computation is carried out,

$$0.73 < \frac{b}{a} < 0.75 \qquad \frac{b}{a} \text{ is approx. } 0.74.)$$

Thus the jet expands with increasing pressure in the reservoir to the limiting value. Its extreme dimensions in width are 0.61 X 2a and 0.74 X 2a where 2a as before denotes the width at the orifice.

An approximate functional formula for the contraction is obtained by making use of the approximate expression for the function x_n given by equation (45), which may be changed to

$$x_n = 1 - \beta s - \frac{\beta s^2}{2} K$$

where

$$K = L + \frac{M}{n + 1} + \frac{N}{2n(1 - \beta s) + 3 + (\beta + 2)s}$$

The coefficients L, M, N do not include the parameter n and have the values

$$L = \frac{\beta}{1 - \beta s} \qquad M = -\frac{2\beta + 1}{1 + (3\beta + 2)s}$$

$$N = 2(2\beta + 1)\left[\frac{1 - \beta s}{1 + (3\beta + 2)s} - \frac{1}{1 - \beta s}\right]$$

If, as before, it is assumed that $\beta = 2.5$, there is obtained finally

$$x_n = 1 - \frac{5s}{2} - \frac{25s^2}{4(2-5s)} + \frac{15s^2}{2+19s}\,\frac{1}{n+1} + \frac{15s^2}{n+\mu}\left[\frac{2}{(2-5s)^2} - \frac{1}{2+19s}\right],$$

where

$$\mu = \frac{6+9s}{4-10s}$$

The series S entering equation (60) readily can be computed by setting

$$S = \sum_1^{\infty}(-1)^{n-1}\frac{n}{4n^2-1}x_n = \left[1 - \frac{5s}{2} - \frac{25s^2}{4(2-5s)}\right]\sum_1^{\infty}(-1)^{n-1}\frac{n}{4n^2-1} + \tag{61}$$

$$+ \frac{15s^2}{2+19s}\sum_1^{\infty}(-1)^{n-1}\frac{n}{(n+1)(4n^2-1)} +$$

$$+ 15s^2\left[\frac{2}{(2-5s)^2} - \frac{1}{2+19s}\right]\sum_1^{\infty}(-1)^{n-1}\frac{n}{(n+\mu)(4n^2-1)},$$

where for simplicity s_0 is replaced by s. The first summation in equation (61) for S is of the form

$$\frac{1}{4}\left[1 + \frac{1}{3} - \frac{1}{3} - \frac{1}{5} + \frac{1}{5} + \frac{1}{7} - \frac{1}{7} - \dots\right] = \frac{1}{4}.$$

It is necessary now to return to the computation of

$$\sigma(\mu) = \sum(-1)^{n-1}\frac{n}{(n+\mu)(4n^2-1)};$$

The particular case of this series corresponding to the value $\mu = 1$ will be the second summation in the equation for S. And $\sigma(\mu)$ may be expressed in the following form:

$$\sigma(\mu) = \int_0^1 \sum(-1)^{n-1}\frac{nt^{n+\mu-1}}{4n^2-1}\,dt.$$

The series under the integral sign can be summed. Thus

$$\sum(-1)^{n-1}\frac{nt^{n+\mu-1}}{4n^2-1} = \frac{t^{\mu-1}}{4}\left[t + \frac{t}{3} - \frac{t^2}{3} - \frac{t^2}{5} + \frac{t^3}{5} + \frac{t^3}{7} - \frac{t^4}{7} - \frac{t^4}{9} + \dots\right].$$

Since the series within the brackets is absolutely convergent for $t < 1$, the order of its terms may be changed and this gives

$$\sum(-1)^{n-1}\frac{nt^{n+\mu-1}}{4n^2-1} = \frac{t^{\mu-\frac{1}{2}}}{4}\left(t^{\frac{1}{2}} - \frac{t^{\frac{3}{2}}}{3} + \frac{t^{\frac{5}{2}}}{5} - \frac{t^{\frac{7}{2}}}{7} + \dots\right) -$$

$$-\frac{t^{\mu-\frac{3}{2}}}{4}\left(t^{\frac{1}{2}}-\frac{t^{\frac{3}{2}}}{3}+\frac{t^{\frac{5}{2}}}{5}-\ldots\right)+\frac{t^{\mu-1}}{4}=\frac{t^{\mu-1}}{4}\left[1-\frac{1-t}{\sqrt{t}}\operatorname{arctg}\sqrt{t}\right].$$

Thus

$$4\sigma(\mu)=4\sum(-1)^{n-1}\frac{n}{(n+\mu)(4n^2-1)}=\frac{1}{\mu}-\int_0^1 t^{\mu-\frac{3}{2}}(1-t)\operatorname{arctg}\sqrt{t}\,dt,$$

or, by integrating by parts,

$$4\sigma(\mu)=\frac{2}{2\mu+1}-\frac{\pi}{4\mu^2-1}+\frac{4\mu}{4\mu^2-1}\int_0^1\frac{t^{\mu-1}dt}{1+t}, \qquad (62)$$

whence

$$4\sigma(1)=\frac{2}{3}-\frac{\pi}{3}+\frac{4}{3}\lg 2=0{,}5437.$$

When the results are added, there is obtained for the contraction

$$\left.\begin{aligned}\frac{b}{a}&=\frac{\pi}{\pi+8S}\\ 8S&=2-5s_0-\frac{25s_0^2}{2(2-5s_0)}+\frac{15s_0^2}{2+19s_0}[1{,}0874-8\sigma(\mu)]+\\ &+\frac{30s_0^2}{(2-5s_0)^2}8\sigma(\mu),\end{aligned}\right\} \qquad (63)$$

where $\mu = \dfrac{6 + 9s_0}{4 - 10s_0}$, $\sigma(\mu)$ is determined by equation (62) and s_0 by the ratio of the pressure in the vessel to that in the free space, $s_0 = \dfrac{\tau_0}{1 - \tau_0}$, which by the formulas of part I is

$$\frac{p_0}{p_1}=(1-\tau_0)^{-1-\beta}=(1+s_0)^{1+\beta}=(1+s_0)^{\frac{7}{2}}.$$

Of greatest interest is the jet contraction for a pressure near the limiting value – that is, for which the velocity of the escaping jet is equal to the velocity of sound propagation in a gas at rest of the same physical state. This limiting pressure corresponds, as has been shown, to the value $\tau_0 = \dfrac{1}{2\beta + 1}$, $s_0 = \dfrac{1}{2\beta}$ and has the value $p_0 = p_1\left(1 + \dfrac{1}{2\beta}\right)^{1+\beta} = 1.89\ p_1$ – that is, 1.89 atmospheres – if the pressure in the free medium is equal to atmospheric. If

$$\mu=\frac{6+9s_0}{4-10s_0}$$

is set in formula $s_0 = \dfrac{1}{2\beta} = 0.2$, the result is $\mu = 3.9$.

To compute accurately the definite integral,

$$j(\mu) = \int_0^1 \frac{t^{\mu-1}dt}{1+t}$$

for such value of μ is rather laborious. In view of the fact, however, that this entire computation is of an approximate character, the problem may be simplified: namely, compute $j(4)$ and $j(3.75)$ and then because of the nearness of the values of these integrals, find $j(3.9)$ by simple interpolation (assuming proportionality between the increment of the function and of the independent variable). There is readily obtained

$$j(4) = \int_0^1 \frac{t^3}{1+t}\,dt = 1 - \frac{1}{2} + \frac{1}{3} - \lg 2 = 0.1402$$

$$j(3.75) = \int_0^1 \frac{t^{11/4}}{1+t}\,dt = 4\left(\frac{1}{3} - \frac{1}{7} + \frac{1}{11}\right)\frac{\pi}{\sqrt{2}} + \sqrt{2}\lg \operatorname{cotg}\frac{\pi}{8}$$

$$= 0.1500$$

whence $j(3.9) = 0.144$; and by equation (62)

$$4\sigma(3.9) = \frac{2}{8.8} - \frac{\pi}{8.8 \times 6.8} + \frac{15.6}{8.8 \times 6.8}\,0.144 = 0.212^*$$

Substituting this value of $\sigma(\mu)$ and the corresponding s_0 in formula (63) yields

$$8S = 0.5 + \frac{3}{29} \times 1.087 + \frac{159}{145} \times 0.424 = 1.08 \qquad (64)$$

the mean value between the limiting values of the series 8S obtained. The contraction then is given by

$$\frac{b}{a} = \frac{\pi}{\pi + 1.08} = 0.74$$

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*The procedure for checking is as follows: Compute the accurate values $4\sigma(4) = 0.2079$ and $4\sigma(3.75) = 0.2193$ whence by interpolation again it is seen that $4\sigma(3.9) = 0.2124$, a value agreeing with that already obtained.

This coefficient decreases with decrease in the pressure in the reservoir because of the increase in value of S. This decrease is sufficiently uniform, as may be seen from the table:

s_0	0,2	0,182	0,154	0,137	0,117
$\frac{p_0}{p_1}$	1,89	1,79	1,65	1,56	1,48
$\frac{b}{a}$	0,74	0,73	0,71	0,70	0,68
μ	3,90	3,50	3,00	2,75	2,50

(65)

Finally, the expression for the quantity of outflowing gas is given by

$$E=2a.\frac{b}{a}\sqrt{2\alpha\tau_0}\rho_0(1+\tau_0)^{\beta},$$

where ρ_0 is the density of the gas in the vessel (at a far-removed region from the orifice) 2a, as before, is the width of the orifice and α is defined by the formula

$$\alpha=\frac{k\gamma}{\gamma-1}\rho_0{}^{\gamma-1},$$

where

$$\gamma=1,40=1+\frac{1}{\beta}.$$

It is necessary first to consider the case of the <u>outflow</u> of a gas from reservoirs with various pressures <u>into a space where the pressure is constant</u> (e.g., into the atmosphere). Then

$$\alpha=\frac{\gamma}{\gamma-1}\,\frac{p_0}{p_1}\,\frac{\rho_1}{\rho_0},\quad 1-\tau_0=\left(\frac{p_1}{p_0}\right)^{\frac{1}{1+\beta}},\quad s_0+1=\left(\frac{p_0}{p_1}\right)^{\frac{1}{1+\beta}},$$

and the final formula

$$E=2a\sqrt{p_1}\,\sqrt{\rho_0}\sqrt{\frac{2\gamma}{\gamma-1}\,\frac{b}{a}}\sqrt{\left(\frac{p_1}{p_0}\right)^{\frac{\beta-1}{\beta+1}}\left[1-\left(\frac{p_1}{p_0}\right)^{\frac{1}{1+\beta}}\right]},\qquad (66)$$

or, again, if $\gamma = 1.40$, $\beta = 2.5$

$$E=2a\sqrt{p_1\rho_0}\,\frac{b}{a}\sqrt{\left(\frac{p_1}{p_0}\right)^{\frac{8}{7}}\left[1-\left(\frac{p_1}{p_0}\right)^{\frac{2}{7}}\right]}.\qquad (66')$$

Since the contraction $\frac{b}{a}$ is a function of s_o and therefore depends on the ratio $\frac{p_o}{p_1}$, with constancy of the ratio, the discharge quantity of the gas is proportional to the square root of the density or inversely proportional to the square root of the temperature.

Now suppose that the state of the gas in the reservoir remains unchanged and consider the flow into a medium of varying pressure. The velocity of sound in the gas corresponding to the same physical conditions at a great distance from the orifice is denoted by c_o. Then

$$c_o = \sqrt{\gamma \frac{p_o}{\rho_o}} = \sqrt{k\gamma\rho_o^{\gamma-1}} = \sqrt{(\gamma - 1)\alpha}$$

and the formula for the discharge may be given as

$$E = 2ac_o \sqrt{\frac{2}{\gamma - 1}}\rho_o \frac{b}{a} \sqrt{\left(\frac{p_1}{p_o}\right)^{\frac{2}{\gamma}} \left[1 - \left(\frac{p_1}{p_o}\right)^{\frac{\gamma - 1}{\gamma}}\right]} \qquad (67)$$

or, by substituting $\gamma = 1.4$,

$$E = 2ac_o\rho_o \sqrt{5}\ \frac{b}{a} \sqrt{\left(\frac{p_1}{p_o}\right)^{\frac{10}{7}} \left[1 - \left(\frac{p_1}{p_o}\right)^{\frac{2}{7}}\right]} \qquad (67')$$

In this formula for E only the last two factors that depend entirely on the pressure ratio vary.

As regards the jet contraction or the <u>discharge coefficient</u> as the magnitude b/a also will be called, it is accurately determined by formula (60) and approximately by formulas (63). For an approximation of accuracy up to 0.01, numerical values have been given of this coefficient for the limiting pressure and several other smaller pressures. It is found that a practical result of the same accuracy is obtained if, instead of the cumbersome formula (63), the following expression is taken for $\frac{b}{a} = \lambda$:

$$\lambda = \frac{b}{a} = \frac{\pi}{\pi + 2 - ks_0} \qquad (68)$$

where k is a certain constant. This formula gives very good values of the contraction for small difference in the pressures between reservoir and free medium - that is, for s near zero; for $s_0 = 0$ the values of b/a by (60) and (68) agree with the Kirchhoff formula for the jet contraction. The constant k is so determined that for $s_0 = \frac{1}{2\beta} = 0.2$ the results by (68) and (63) agree. For this it is necessary that

$$(2 - k)(0.2 = 8S(0.2))$$

and, since $8S$ has been found (in equation (64)) equal to 1.08, $k = 4.6$. The values of $8S$ for the values of s_0 given in table (65) are correspondingly equal to 1.08; 1.16; 1.28; 1.35; 1.44. To these correspond

1.08; 1.17; 1.29; 1.37; 1.46

numerical values of the binomial $2 - 4.6s_0$ entering formula (68) in place of 8S.

Such difference has no effect on the results for the second decimal accuracy which has been assumed.

The agreement will be even better if the discharge coefficient is expressed by the formula

$$\lambda = \frac{b}{a} = \frac{\pi}{\pi + 2 - 5s_0 + 2s_0^2} \qquad (69)$$

The series of values of the function $2 - 5s_0 + 2s_0^2$ for the same s_0 will, to an accuracy of 0.01, be equal to 1.08; 1.16; 1.28; 1.35; and 1.44, which are equal, respectively, to the above obtained approximate values of $8S$ entering the exact contraction formula. Thus, formula (69) or (68) for $k = 4.6$ quite well expresses the function λ and may very conveniently be applied for practical purposes. The discharge formula (67) for the assumed round values of $\gamma = 1.4$ and $\beta = 1/(\gamma - 1) = 2.5$ tak a sufficiently simple form if the variable s_0 is introduced in it. The latter, as has been shown, is connected with the pressure ratio p_0/p_1:

$$s_0=\left(\frac{p_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}}-1=\left(\frac{p_0}{p_1}\right)^{\frac{2}{7}}-1; \qquad (70)$$

By transforming equation (67) in this manner, the discharge formula is obtained as follows:

$$E=2ac_0\rho_0\lambda\frac{\sqrt{5s_0}}{(1+s_0)^3}\cdot \qquad (71)$$

It is noted that of the two simplified formulas for λ the second, (69), is the more rational. Thus, to turn for a moment to formula (60)

$$\lambda=\frac{\pi}{\pi+8\sum(-1)^{n-1}\frac{n}{4n^2-1}x_{n,0}}$$

and substitute in it for $x_{n,0}$ the trinomial

$$x_n(0)+s_0x'_n(0)+\frac{s_0^2}{2}x''_n(0),$$

expresses approximately the function x_n. From the equation for x_n

$$x'_ns(1+s)+x_n\beta s+nx_n^2-n(1-2\beta s)=0$$

which gives for $x_n(0)=1$, $x'_n(0)=-\beta$

$$x''_n(0)=-\beta\frac{n\beta-\beta-1}{n+1}=-\beta^2+\beta\frac{2\beta+1}{n+1}.$$

Substitute these values in the above trinomial and compute the series S, entering the denominator of the formula for λ:

$$S=\sum(-1)^{n-1}\frac{n}{4n^2-1}\left(1-\beta s_0-\frac{\beta^2}{2}s_0^2\right)+\beta\frac{2\beta+1}{2}s_0^2\sum(-1)^{n-1}\frac{n}{(4n^2-1)(n+1)};$$

but

$$\sum(-1)^{n-1}\frac{n}{4n^2-1}=\frac{1}{4};$$

$$\sum(-1)^{n-1}\frac{n}{(4n^2-1)(n+1)}=\int_0^1\sum(-1)^{n-1}\frac{nt^n}{4n^2-1}dt=\int_0^1\left(1+\frac{t-1}{\sqrt{t}}\operatorname{arctg}\sqrt{t}\right)dt,$$

or carry out the integration

$$4\sum(-1)^{n-1}\frac{n}{(4n^2-1)(n+1)}=\frac{4}{3}\lg 2-\frac{\pi-2}{3}=0,5437;$$

whence

$$8S=2\left(1-\beta s_0-\frac{\beta^2}{2}s_0^2\right)+\beta(2\beta+1)0,5437s_0^2$$

Assume as before $\beta = 2.5$, which gives

$$8S=2-5s_0+1,9s_0^2$$

and

$$\lambda=\frac{\pi}{\pi+2-5s_0+1,9s_0^2}.$$

Limiting to an accuracy of 0.01, consider this formula identical with (69).

It is considered of interest, finally, to call the reader's attention to a very simple connection between the variable s_0 by which all the characteristic constants of the problem and the temperature of the jet is expressed. For the density and the pressure within the gas flow the formulas are:

$$\rho = \rho_0 (1-\tau)^{\beta}$$

$$p = k\rho^{\gamma} = k\rho_0{}^{\gamma}(1-\tau)^{\beta\gamma} = p_0(1-\tau)^{\beta\gamma}$$

hence

$$\frac{p}{\rho} = \frac{p_0}{\rho_0}(1-\tau)^{\beta(\gamma-1)} = \frac{p_0}{\rho_0}(1-\tau)$$

and since by Mariotte and Gay-Lussac's law $p/\rho = RT$, where T is the absolute temperature at the point considered, the foregoing equation may be rewritten as

$$\frac{T}{T_0} = 1-\tau = \frac{1}{1+s}$$

since $s = \dfrac{\tau}{1-\tau}$ (see pt. II). Applying this relation to the part of the jet remote from the orifice and denoting the temperature of the gas there by T_1 yields

$$\tau_0 = \frac{T_0 - T_1}{T_0}, \qquad s_0 = \frac{T_0 - T_1}{T_1} \tag{71'}$$

This investigation on the outflow of gases will be supplemented by comparing the results obtained on the one hand with approximate theoretical formulas applied for computing the discharge and on the other hand with the results of tests. Purely empirical formulas are not dealt with although some of the latter well express the phenomenon within certain limits, as, for example, the formula of Parenty (reference 8).

For a rational basis of the approximate theoretical treatment the adiabatic law was assumed (also in this investigation) the assumption being made that the outflowing jet at a certain distance from the orifice has the maximum contraction and that at the points of this contraction the velocity of the gas particles is constant. As a result the following formula is obtained for the discharge formula:

$$E = Sc_o\rho_o\lambda\sqrt{\frac{2}{\gamma-1}}\sqrt{\left(\frac{p_1}{p_o}\right)^{\frac{2}{\gamma}}\left[1-\left(\frac{p_1}{p_o}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{67a}$$

where S is the orifice area and λ the <u>discharge coefficient</u> equal to the ratio of the area of the contracted cross-section to the area of the orifice.

The above equation does not differ in form from equation (67), the only difference being that the discharge coefficient was not determined for any, or even for a particular shape of orifice. It has usually been assumed that it has a constant value depending only on the shape of the vessel and orifice. Such assumption, as is seen from the problem solved here, is far from true. In this case this coefficient, for a change in $\frac{p_1}{p_o}$ from 1 to the limiting value 0.53, increases from 0.61 to 0.74. The increment thus constitutes more than 21 percent of the lower limiting value. If the orifice were round and not in the form of a slit, as in this case, a still sharper difference in the values of λ should be expected, for then the lines of flow would converge toward the orifice from all azimuths and not from two as is true in the present case. For this reason, when it was attempted to apply the discharge formula with constant λ to the determination of the true discharge, experiment did not turn out to be in agreement with the theory. In view of this Parenty (reference 8) relying on the tests of Hirn (reference 9) assumed that to apply the formulas based on the adiabatic law of pressure change to flow discharges from orifices was incorrect. However the results of Hirn's tests which he presents show precisely the increase in the discharge coefficient λ which is predicted by the present theory. The possibility of such a variation was foreseen by Parenty but having remarked on it gave it no further consideration since he had no means of making a quantitative estimate of the increase in λ.

Another fact is considered here that is of interest. Having obtained the discharge formula (67a) Saint-Venant called attention to the following paradox. If this formula were applied for any ratio of pressures in the reservoir and the open medium, the discharge, increasing from zero, would pass through a maximum for a certain pressure ratio and thereafter should again decrease, becoming zero at $p_1/p_0 = 0$. The value of P for p_1/p_0 corresponding to the maximum discharge is determined from the condition

$$\frac{\partial \frac{p_1}{p_0}^{\frac{1}{\gamma}}\left[1-\left(\frac{p_1}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{1}{2}}}{\partial p_1} = 0$$

hence

$$P = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} = \left(1 - \frac{1}{2\beta+1}\right)^{\beta+1} = 0.53$$

This is just the limiting pressure ratio corresponding to the instant when the gas in the contracted part of the jet moves with the velocity of sound propagation at that point, as remarked by Hougoniot (reference 10). This condition cannot of course occur in practice. When Saint-Venant conducted his tests on the flow of gases he found that on lowering the pressure in the free medium and varying the ratio p_1/p_0 from 1 to 0.53 the discharge increases; but on further lowering p_1 the process becomes regular, there being no further increase in the discharge. This surprising result was long looked upon with doubt but Hirn's tests, conducted not very long ago, confirmed the results of Saint-Venant with the difference, however, that Hirn observed an increase in the discharge beyond the limit indicated by his predecessor. According to Hirn's tests the discharge reaches the maximum value for $p_1/p_0 = 0.26$, approximately. The change in discharge on lowering p_1/p_0 from 0.53 to 0.26 is, however, insignificant, for which reason this may not have been noted by Saint Venant in his less detailed observations.

If it was attempted to apply formula (70) for λ determined by relation (69) beyond the proper limits of its applicability the same paradoxical result would be obtained except that the maximum discharge would correspond to a value of p_1/p_0 somewhat less than 0.27, a value very close to Hirn's limit. This interesting agreement shows that the present formula expresses sufficiently well the investigated phenomenon in its essential features.

It is now natural to inquire into the character of the motion in the case where the pressure in the free medium into which the jet discharges is lower than the limiting, that is, if $\frac{p_1}{p_0} < 0.53$. If it is assumed that flow remains steady with continuous change in the velocity and pressure within the boundaries of the moving gas mass the region of the variables τ, θ would be in the form of a semicircle of radius $> \frac{1}{2\beta+1}$. This is the very region considered in part I where it was shown that steady motion of the type that is of interest to us was not included in the number of possible motions. Hougoniot(reference 10), states the following: if $p_1 < 0.53\ p_0$ the escaping jet is divided by the surface over which the velocity of the particles is equal to the velocity of sound, into two parts, the pressure in passing through this surface changing discontinuously; above this partition surface in the jet the pressure is equal to $0.53\ p_0$ and below it is equal to p_1. (This phenomenon reminds Parenty of the separation from solid bodies.) But the flow of the gas is considered as steady in both parts of the escaping jet and the surface of pressure discontinuity as everywhere normal to the streamlines.

This latter supposition appears highly improbable since the character of the motion in the upper part of the flow should radically change immediately after the pressure in the free medium passes beyond the limit of $0.53\ p_0$. In fact, first of all it can be easily shown that the width of each elementary tube of flow will be a minimum at the point where the limiting pressure occurs. This is because the cross-section is determined as the ratio of the quantity of gas carried by the tube divided by $\rho_0 \sqrt{2a\tau_0}(1-\tau_0)^{\beta}$ and this denominator passes through a maximum at $\tau = \frac{1}{2\beta+1}$. Therefore taking the tubes of flow normal to the line $\tau = \frac{1}{2\beta+1}$ a minimum discharge of gas from the vessel shall be obtained for the case where this line is a segment enclosing the orifice. The discharge coefficient will then evidently equal 1 and, therefore in passing through the limiting pressure in the receiver this coefficient and the gas discharge should immediately increase by more than 30 percent, a condition that is in entire disagreement with reality.

It is assumed on the other hand that the phenomenon could be explained in the following way. Together with the authors referred to it is supposed that the jet is divided by a certain boundary surface on passing through which the pressure changes very sharply. It may be imagined that the trace of this surface on a plane parallel to the flow as a curve supported at the edge of the orifice further on the curve resembling the contour of a tongue of flame moves into the open medium. Above this limit (inside the vessel and the adjoining part of the jet) the flow will be stable and the pressure drops from p_o at the far removed parts of the vessel to $0.53p_o$ on the described boundary curve. At the remote part, however, the jet forms waves.* These waves have an enveloping boundary curve. In a very thin layer of this part of the jet adjoining the curve the mean pressure will be $< 0.53p_o$ and the velocity of propagation of sound $c_2 < c_1$ being the same velocity for the boundary layer lying beyond the boundary curve. The lowered pressure tends to be propagated beyond the boundary curve, following along the jet in the form of a plane wave. But this wave is carried backward by each infinitely thin jet element and since the velocity of the gas particles is also c_1 no waves are observed in the upper parts of the gas flow. In order that the boundary curves may serve as an envelope of the waves approaching it, it is sufficient, as it appears to assume that the velocity of the waves normal to this line is the same whether the wave moves upward or downward. If λ denotes the angle formed by a jet element with the boundary curve passing through a given point, then having determined both normal velocities by Riemann's rule equating them and applying very simple hydrodynamic considerations there is obtained

*These waves have been observed and studied recently by Emden. The results of his tests are described in reference 4. The waves appear immediately after the pressure in the reservoir drops below $0.53p_o$; their length increases with the lowering of the pressure in the reservoir. Emden also gives a theory of the phenomenon which, however, is entirely unfounded. It is sufficient to say that notwithstanding the existence of waves Emden considers the pressure throughout the jet as constant, which of course is impossible, and makes this assumption the basis of his analysis.

$$\sin\lambda = \frac{c_1 - c_2}{c_1\left(1+\frac{\rho_1}{\rho_2}\right)}$$

If the pressure and density varied, even very sharply, but continuously, $\rho_1 : \rho_2 = 1$. Thus it is seen that <u>the tubes of flow intersect the boundary curve at a constant angle</u>.

With this hypothesis steady flow above the boundary curve may be determined strictly mathematical regardless of what occurs in the remaining part of the jet. It is not difficult to show; namely, that on this curve, given by equation $\tau = \tau_0 = \frac{1}{2\beta+1}$ the relation holds

$$\varphi \, tg\lambda(1-\tau_0)^{\beta} \pm \psi = Q$$

one sign corresponding to the left half of the boundary curve, the other to the right half and ψ and φ denoting, as before, the stream function and velocity potential. Including this relation among the boundary conditions it may be shown next that, together with the other conditions, it is entirely sufficient for the determination of φ and ψ in the τ, θ region. Having found φ and ψ it is easy to determine the gas discharge per second. It appears that if this discharge were strictly constant or changing slightly with change in λ from zero to its limiting value, the explanation just given would be near the truth. Incidently it may be said that the limits within which λ may vary are not wide; this angle will not be large. For this reason the relation previously given between ψ and φ in all probability will give a result not deviating too much from that which would be obtained by simply taking $\psi = \pm Q$ along the boundary curve. A small variation in the discharge may also be expected from the consideration that its value will depend on $\int \sin\lambda ds$ extended over the boundary curve. This integral is evidently equal to the total length of this curve multiplied by $\sin\lambda$, and its length will decrease with increasing λ. It may be noted, finally, that in assuming the above explanation of the flow phenomena there is obtained an entirely continuous transition from the problem solved above to those cases where the given analysis is inapplicable. A mathematical treatment of the proposed hypothesis is intended in the near future.

PART IV

PRESSURE OF A GAS JET ON A PLATE

The study of pressure of a gas jet on a plate will begin with the consideration of the impact of a gas jet on a plate perpendicular to the initial direction of the jet, assuming that the jet is symmetrically divided into two parts by the plate. Again the corresponding problem for the case of an incompressible liquid is used. The solution of this problem is given in the paper by Joukowsky (reference 2). By use of the same variables as in part III,

$$w = \varphi_1 + i\psi_1$$

$$z = x + iy, \qquad \lg v_0 \frac{dz}{dw} = \vartheta + i\theta$$

$$\varphi_1 + i\psi_1 = -\frac{Q}{2\pi} \lg \left[1 - \frac{\sin^2 m}{\sin^2(\theta - i\vartheta)}\right] \tag{71}$$

where m is the angle of inclination with the X axis at distant points of the two parts into which the jet is divided by the plate. For the X axis the line of symmetry of the jet is taken, the initial direction of the jet being in this case parallel to the X axis.

It will be shown that formula (71) expresses precisely the required liquid flow. Attention will be directed first to the range of complex variables w and $\vartheta + i\theta$ which correspond to the flow sketched in figure 5. The region w is bounded by two straight lines parallel to the real axis, symmetrically placed with respect to it at a distance $\frac{Q}{2}$. In the sketch the outer boundaries of the jet EA' and DC' correspond to these straight lines. The flow boundaries CBO and AYO correspond to the upper and lower sides of the positive part of the real axis of the region w; the point O corresponds to $w = 0$.

The region $\vartheta + i\theta$ is bounded, in the first place, by the segments of the straight lines parallel to the axis and having coördinates $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$ lying to the right of the imaginary axis; in the second place, by the segment of the imaginary axis lying between the above-mentioned parallels. On the sketch the straight line $\theta = \frac{\pi}{2}$ corresponds to the right-hand part of the plane and the line $\theta = -\frac{\pi}{2}$ to the left-hand part. The segment of the imaginary axis included between the points $\theta = \frac{\pi}{2}$ and $\theta = m$ corresponds to the boundary YA of the flow, for here the velocity is v_0, $\vartheta = 0$. The segment symmetrical to that just mentioned corresponds to the curve BC. Finally, the boundary C'D is represented in the $\vartheta + i\theta$ region by the segment included between the points $\theta = -m$ and $\theta = 0$ of the imaginary axis and the curve EA' by the segment bounded by the points $\theta = 0$ and $\theta = m$.

Now, proceeding along the boundaries of the $\vartheta + i\theta$ region, the author will show that the point w will then describe the above-mentioned boundary of the w region. With the point $\theta = \frac{\pi}{2}$, $\vartheta = \infty$ as the starting place, it can be seen from equation (71) that for these values of ϑ and θ, $w = 2k\pi i$, where k is an arbitrary integer - it will be taken equal to zero. If now the point $\vartheta + i\theta$ moves along the line $\theta = \frac{\pi}{2}$, then w moves along its real axis at the upper side of this axis, since for $\theta = \frac{\pi}{2} - \epsilon$, then, at infinitely small ϵ, $\psi_1 = k\epsilon$, where k is some positive quantity. When the point $\vartheta + i\theta$ arrives at the position $\vartheta = 0$, $\theta = \frac{\pi}{2}$, w will have passed along the segment of the real axis from 0 to $\varphi_1 = -\frac{Q}{\pi} \lg(1 - \sin^2 m)$. As $\vartheta + i\theta$ moves farther along the imaginary axis, the point w will continue its motion along the φ_1 axis in the same direction up to $\varphi_1 = \infty$ corresponding to $\vartheta = 0$, $\theta = m$. In passing through the point $\vartheta = 0$, $\theta = m$, the logarithm in formula (71) receives an increment $-\pi i$ and for $\vartheta = 0$, $m > \theta > 0$, w will move forward along the straight line $w = i\frac{Q}{2}$ from $+\infty + i\frac{Q}{2}$ to $-\infty + i\frac{Q}{2}$; this position w corresponds to $\theta = 0$. In passing through the point $\vartheta = 0$, $\theta = 0$, the logarithm of formula (71) will receive an increment $2\pi i$ and w will pass discontinuously from the upper boundary of its region to the lower and will move along it from $\overline{-\infty} - i\frac{Q}{2}$ to $+\infty - i\frac{Q}{2}$ as $\vartheta + i\theta$ moves from 0 to $-im$. Further, as $\vartheta + i\theta$ moves through $-im$, the logarithm increases by $-\pi i$, w jumps to the point $+\infty$ and along the lower side of the positive part of the φ_1 axis returns to its initial position as $\vartheta + i\theta$ moves from $-im$ to $-i\frac{\pi}{2}$ and from $-i\frac{\pi}{2}$ to $\infty - i\frac{\pi}{2}$ along the boundaries of its region.

Thus the fact is shown that formula (71) is an actual solution of the problem of the impact of a liquid stream on a plate. In order to solve this problem for the gas jet, it is necessary to proceed according to the rule given. The expression w is expanded into a series and its imaginary part separated.

Thus,

$$\frac{2\pi w}{Q} = 2 \lg \sin \frac{\vartheta + i\theta}{i} - \lg \left(\sin^2 \frac{\vartheta + i\theta}{i} - \sin^2 m\right)$$

$$= 2 \lg \sin \frac{\vartheta + i\theta}{i} - \lg \left(\cos 2m - \cos 2\frac{\vartheta + i\theta}{i}\right) + \lg 2$$

$$= 2 \lg \sin \frac{\vartheta + i\theta}{i} - \lg \sin \frac{\vartheta + i\theta - im}{i} - \lg \sin \frac{\vartheta + i\theta + im}{i}$$

Introducing exponential functions in place of the trigonometrical transforms this into

$$\frac{2\pi w}{Q} = 2\lg(1-e^{-2\vartheta-2i\theta}) - \lg(1-e^{-2\vartheta-2i(\theta-m)}) -$$

$$-\lg(1-e^{-2\vartheta-2i(\theta+m)}) = \sum_{n=1}^{\infty}\frac{e^{-2n\vartheta}}{n}\Big\{\cos 2n(\theta-m) - i\sin 2n(\theta-m) +$$

$$\cos 2n(\theta+m) - i\sin 2n(\theta+m) - 2\cos 2n\theta + 2i\sin 2n\theta\Big\},$$

or, after reducing,

$$\frac{2\pi w}{Q} = -2\sum_{1}^{\infty}\frac{e^{-2n\vartheta}}{n}(1-\mathrm{co}2nm)(\cos 2n\theta - i\sin 2n\theta),$$

and, finally, for ψ_1

$$\frac{\pi\psi_1}{Q} = \sum_{1}^{\infty}\frac{e^{-2n\vartheta}}{n}\sin 2n\theta(1-\cos 2nm).$$

Since $e^{-\vartheta} = \frac{v}{v_0} = \sqrt{\frac{\tau}{\tau_0}}$, $(v = \sqrt{2\alpha\tau})$

$$\frac{\pi\psi_1}{Q} = \sum_{1}^{\infty}\frac{1}{n}\left(\frac{\tau}{\tau_0}\right)^n \sin 2n\theta(1-\cos 2nm).$$

This is the expression of the stream function in the variables τ, θ for the liquid flow. Hence for the gas flow the same problem should be solved by the formula

$$\frac{\pi\psi}{Q} = \sum_{1}^{\infty}\frac{1}{n}\left(\frac{\tau}{\tau_0}\right)^n \frac{y_n}{y_{n,0}}\sin 2n\theta(1-\cos 2nm). \qquad (72)$$

According to formula (24) of part I the following expression is obtained for φ:

$$\frac{\pi\varphi}{Q} = C-(1-\tau)^{-\beta}\sum_{1}^{\infty}\frac{1}{n}\left(\frac{\tau}{\tau_0}\right)^n \frac{y_n}{y_{n,0}}x_n\cos 2n\theta(1-\cos 2nm). \qquad (73)$$

If it is assumed that $C = 0$, then, as in the case of the liquid flow, $\tau = \theta = 0$, $\varphi = 0$, and $\psi = 0$.

The series for φ and ψ are absolutely and uniformly convergent for any $\tau < \tau_0$ as is clear from the general considerations of part II, and therefore the problem is solved by the relations (72) and (73).

The width of the plate and the pressure on it will now be determined. The width of the plate will be denoted by $2l$. Then l is found by substituting in the expression for the coordinate y the values $\theta = \frac{\pi}{2}$, $\tau = \tau_0$.

But

$$dy = \frac{\partial y}{\partial \varphi} d\varphi + \frac{\partial y}{\partial \psi} d\psi, \qquad \Big/^{\theta=\frac{\pi}{2}} y = \left(\int dy\right)_{\theta=\frac{\pi}{2}}$$

and, since for $\theta = \frac{\pi}{2}$, $\psi = 0 =$ constant

$$(dy)_{\theta=\frac{\pi}{2}} = \frac{\partial y}{\partial \varphi} d\varphi = \frac{\partial y}{\partial \varphi}\frac{\partial \varphi}{\partial \tau} d\tau = \Big/^{\theta=\frac{\pi}{2}} \frac{\sin\theta}{\sqrt{2\alpha\tau}} \frac{\partial \varphi}{\partial \tau} d\tau$$

By substituting the formula obtained for φ and noting that

$$l = \Big/^{\tau=\tau_0,\ \theta=\frac{\pi}{2}} y$$

there is obtained

$$l = \frac{Q}{\pi\sqrt{2\alpha}} \sum_{n=1}^{\infty}\left\{(-1)^{n-1} \frac{1 - \cos 2nm}{n\tau_0^{\,n}\, y_{n,0}} \int_0^{\tau_0} \frac{d}{d\tau} (1 - \tau)^{-\beta} \left(\tau^n y_n + \frac{\tau^{n+1}}{n} y'_n\right)\frac{d\tau}{\sqrt{\tau}}\right\} \qquad (74)$$

In computing the integrals entering this series, it is noted first of all that

$$\frac{d}{d\tau}(1-\tau)^{-\beta}\tau\left(\tau^{n-1}y_n+\frac{\tau^n}{n}y'_n\right)=\frac{1}{n}\frac{d}{d\tau}\tau(1-\tau)^{-\beta}z'_n$$

and on the basis of the equation for z_n

$$\frac{d}{d\tau}\tau(1-\tau)^{-\beta}z'_n=n^2[1-(2\beta+1)\tau](1-\tau)^{-\beta-1}\frac{z_n}{\tau}$$

Hence

$$J(\tau_0)=\int_0^{\tau_0}\frac{d}{d\tau}(1-\tau)^{-\beta}\tau\left(\tau^{n-1}y_n+\frac{\tau^n}{n}y'_n\right)\frac{d\tau}{\sqrt{\tau}}=$$

$$=\frac{1}{n}\int_0^{\tau_0}\frac{d\tau}{\sqrt{\tau}}\frac{d}{d\tau}\tau(1-\tau)^{-\beta}z'_n=\frac{1}{n}\sqrt{\tau_0}(1-\tau_0)^{-\beta}z'_{n,0}+ \qquad (75)$$

$$+\frac{1}{2n}\int_0^{\tau_0}\frac{(1-\tau)^{-\beta}z'_n}{\sqrt{\tau}}d\tau.$$

On the other hand

$$\int_0^{\tau_0}\frac{(1-\tau)^{-\beta}z_n}{\sqrt{\tau}}d\tau=\frac{(1-\tau_0)^{-\beta}}{\sqrt{\tau_0}}z_{n,0}+$$

$$+\frac{1}{2}\int_0^{\tau_0}\frac{1-(2\beta+1)\tau}{\tau\sqrt{\tau}}(1-\tau)^{-\beta-1}z_n d\tau;$$

and since on the basis of the differential equation for the function z_n

$$\frac{1-(2\beta+1)\tau}{\tau}(1-\tau)^{-\beta-1}z_n=\frac{1}{n^2}\frac{d}{d\tau}\tau(1-\tau)^{-\beta}z'_n,$$

therefore

$$\int_0^{\tau_0}\frac{(1-\tau)^{-\beta}z'_n}{\sqrt{\tau}}d\tau=\frac{(1-\tau_0)^{-\beta}}{\sqrt{\tau_0}}z_{n,0}+\frac{1}{2n^2}\int_0^{\tau_0}\frac{d\tau}{\sqrt{\tau}}\frac{d}{d\tau}\tau(1-\tau)^{-\beta}z'_n=$$

$$=\frac{(1-\tau_0)^{-\beta}}{\sqrt{\tau_0}}z_{n,0}+\frac{1}{2n}J(\tau_0).$$

Comparing this relation with (75) results in

$$\frac{4n^2-1}{4n}J(\tau_0)=(1-\tau_0)^{-\beta}\left(\frac{z_{n,0}}{2\sqrt{\tau_0}}+\sqrt{\tau_0}z'_{n,0}\right), \qquad (76)$$

or

$$J(\tau)=\frac{4n}{4n^2-1}(1-\tau)^{-\beta}\frac{d}{d\tau}z_n\sqrt{\tau}=\frac{4n}{4n^2-1}(1-\tau)^{-\beta}z_n\sqrt{\tau}\frac{d}{d\tau}\lg z_n\sqrt{\tau};$$

Since $z_n = \tau^n y_n$, $1+\frac{\tau}{n}\frac{y'_n}{y_n}=x_n$,

$$J(\tau)=\frac{2n}{4n^2-1}\tau^{n-\frac{1}{2}}y_n(1+2nx_n)(1-\tau)^{-\beta} \qquad (76')$$

With the aid of formula (76'), equation (74) can be written in the form

$$l=\frac{2Q}{\pi\sqrt{2\alpha\tau_0}}\left\{\sum_1^\infty(-1)^{n-1}\frac{1-\cos 2nm}{4n^2-1}+\right.$$

$$\left.+\sum_1^\infty(-1)^{n-1}\frac{1-\cos 2nm}{4n^2-1}\,2nx_{n,0}\right\}(1-\tau_0)^{-\frac{3}{2}}. \tag{77}$$

The first of the summations entering the above formula is an absolutely convergent series. It is computed as follows:

$$\sum_1^\infty(-1)^{n-1}\frac{1-\cos 2nm}{4n^2-1}=\sum_1^\infty(-1)^{n-1}\frac{1}{4n^2-1}-$$

$$-\sum_1^\infty(-1)^{n-1}\frac{\cos 2nm}{4n^2-1};$$

$$\sum_1^\infty(-1)^{n-1}\frac{1}{4n^2-1}=\frac{1}{2}\left(1-\frac{1}{3}-\frac{1}{3}+\frac{1}{5}+\frac{1}{5}-\frac{1}{7}-\right.$$

$$\left.-\frac{1}{7}+\frac{1}{9}+\ldots\right)=\frac{1}{2}\left(\frac{\pi}{2}-1\right).$$

$$\sum_1^\infty(-1)^{n-1}\frac{\cos 2nm}{4n^2-1}=\cos m\sum_1^\infty(-1)^{n-1}\frac{\cos(2n-1)m}{4n^2-1}-$$

$$-\sin m\sum_1^\infty(-1)^{n-1}\frac{\sin(2n-1)m}{4n^2-1}; \tag{78}$$

It is readily seen, however, that

$$K=\sum_1^\infty(-1)^{n-1}\frac{\cos(2n-1)m-i\sin(2n-1)m}{4n^2-1}=$$

$$=\int_0^1\sum_1^\infty(-1)^{n-1}\frac{t^{2n}}{2n-1}\left[\cos(2n-1)m-i\sin(2n-1)m\right]dt=$$

$$=\int_0^1 t\sum_1^\infty(-1)^{n-1}\frac{\zeta^{2n-1}}{2n-1}\,dt,$$

where $\zeta = te^{-im}$

Since $\sum_{1}^{\infty}(-1)^{n-1}\frac{\zeta^{2n-1}}{2n-1}=\operatorname{arctg}\zeta$, the integral obtained is expressed as follows:

$$K=\int_0^1 t\operatorname{arctg} te^{-im}dt=\frac{1}{2}\operatorname{arctg}e^{-im}-\frac{1}{2}\int_0^1\frac{t^2e^{-im}dt}{1+t^2e^{-2im}}=$$

$$=\frac{1}{2}\operatorname{arctg}e^{-im}-\frac{e^{im}}{2}+\frac{e^{im}}{2}\int_0^1\frac{dt}{1+t^2e^{-2im}}=$$

$$=\frac{1}{2}(1+e^{2im})\operatorname{arctg}e^{-im}-\frac{e^{im}}{2};$$

and since

$$\operatorname{arctg}e^{-im}=\frac{\pi}{4}-\frac{i}{2}\lg\sqrt{\frac{1+\sin m}{1-\sin m}},$$

therefore

$$K=\frac{\pi}{8}(1+\cos 2m)-\frac{\cos m}{2}+\frac{\sin 2m}{8}\lg\frac{1+\sin m}{1-\sin m}+$$

$$+i\left(\frac{\pi}{8}\sin 2m-\frac{\sin m}{2}+\frac{1+\cos 2m}{8}\lg\frac{1+\sin m}{1-\sin m}\right).$$

Returning to the initial formula for K, it is seen that the first row of its new expression gives the expression in finite form of the function $\sum_{1}^{\infty}(-1)^{n-1}\frac{\cos(2n-1)m}{4n^2-1}$, and the coefficient of i in the second row is equal to $-\sum_{1}^{\infty}(-1)^{n-1}\frac{\sin(2n-1)m}{4n^2-1}$; then (78) yields

$$\sum_{1}^{\infty}(-1)^{n-1}\frac{\cos 2nm}{4n^2-1}=\frac{1}{2}\left(\frac{\pi}{2}\cos m-1\right),$$

and, finally,

$$\sum_{1}^{\infty}(-1)^{n-1}\frac{1-\cos 2nm}{4n^2-1}=\frac{\pi}{4}\left(1-\cos m\right). \qquad (79)$$

Making use of this formula, equation (77) is finally transformed into the following relation

$$l=\frac{Q}{\pi\sqrt{2\alpha\tau_0}}\left\{\frac{\pi}{2}(1-\cos m)+\right. \tag{80}$$

$$\left.+\sum_1^\infty(-1)^{n-1}\frac{4nx_{n,0}}{4n^2-1}(1-\cos 2nm)\right\}(1-\tau_0)^{-\beta}.$$

The series in the above formula must be convergent. In order to prove this the remainder term is set up, making use of formula (49) of part II for x_n. There results

$$x_{n,0}=\sqrt{1-2\beta s_0}+k\lambda_n\frac{s_0}{n^{\frac{4}{3}}},$$

$$R_n=\sqrt{1-2\beta s_0}\sum(-1)^{n-1}\frac{4n}{4n^2-1}(1-\cos 2nm)+$$

$$+Ks_0\sum\frac{(-1)^{n-1}(1-\cos 2nm)\lambda'_n}{n^{\frac{4}{3}}};$$

It is clear that the $\lim_{n=\infty} R_n = 0$, since λ_n and λ_n' are proper fractions.

By passing to the computation of R - the resultant pressure on the plate - the pressure behind it is denoted by p_1 and it is noted that

$$R=2\int_0^l(p-p_1)dy=2\int_0^{\tau_0}\left\{\frac{p}{\sqrt{2\alpha\tau}}\frac{\partial\varphi}{\partial\tau}d\tau\right\}_{\theta=\frac{\pi}{2}}-2p_1l;$$

But

$$p=k\rho^\gamma=k\rho_0{}^\gamma(1-\tau)^{\beta+1},\ \text{since}\ \rho=\rho_0(1-\tau)^\beta,$$

hence the integral entering the expression for R - it is denoted by T - on substituting the value of the function φ, becomes

$$T=\frac{\pi k\rho_0{}^\gamma}{Q\sqrt{2\alpha}}\sum_1^\infty\left\{(-1)^{n-1}\frac{1-\cos 2nm}{n\tau_0{}^n y_{n,0}}\times\right.$$

$$\left.\times\int_0^{\tau_0}(1-\tau)^{\beta+1}\frac{d}{d\tau}(1-\tau)^{-\beta}\left(\tau^n y_n+\frac{\tau^{n+1}}{n}y'_n\right)\frac{d\tau}{\sqrt{\tau}}\right\}.$$

Further, using the expression

$$J(\tau)=\int_0^\tau\frac{d}{d\tau}(1-\tau)^{-\beta}\left(\tau^n y_n+\frac{\tau^{n+1}}{n}y'_n\right)\frac{d\tau}{\sqrt{\tau}},$$

there is obtained

$$S(\tau_0)=\int_0^{\tau_0}(1-\tau)^{\beta+1}\frac{d}{d\tau}(1-\tau)^{-\beta}\left(\tau^n y_n+\frac{\tau^{n+1}}{n}y'_n\right)\frac{d\tau}{\sqrt{\tau}}=$$

$$=\int_0^{\tau_0}(1-\tau)^{\beta+1}dJ=(1-\tau_0)^{\beta+1}J(\tau_0)+(\beta+1)\int_0^{\tau_0}(1-\tau)^{\beta}J(\tau)d\tau;$$

By (76)

$$J(\tau)=\frac{4n}{4n^2-1}(1-\tau)^{-\beta}\frac{d}{d\tau}z_n\sqrt{\tau};$$

hence

$$S(\tau_0)=(1-\tau_0)^{\beta+1}J(\tau_0)+\frac{4n(\beta+1)}{4n^2-1}z_{n,0}\sqrt{\tau_0}.$$

Substituting this expression in the sum to be computed, yields

$$T=\frac{\pi k\rho_0{}^{\gamma}(1-\tau_0)^{\beta+1}}{Q\sqrt{2\alpha}}\sum_1^{\infty}(-1)^{n-1}\frac{1-\cos 2nm}{n\tau_0^n y_{n,0}}J(\tau_0)+$$

$$+4\frac{(\beta+1)\pi k\rho_0{}^{\gamma}}{Q\sqrt{2\alpha}}\sqrt{\tau_0}\sum_1^{\infty}(-1)^{n-1}\frac{1-\cos 2nm}{4n^2-1};$$

The first of these terms on the basis of (74) gives the magnitude

$$k\rho_0{}^{\gamma}(1-\tau_0)^{\beta+1}l=p_1 l;$$

and therefore making use of (79) and noting that

$$k\rho_0{}^{\gamma}(1+\beta)=\frac{k\gamma}{\gamma-1}\rho_0{}^{\gamma}=\alpha\rho_0,$$

there is obtained

$$T=p_1 l+\frac{Q\rho_0\sqrt{\alpha\tau_0}}{\sqrt{2}}(1-\cos m).$$

The above formula for R yields

$$R=Q\rho_0\sqrt{2\alpha\tau_0}(1-\cos m). \qquad (81)$$

By making use of equation (80) and considering the relation $\sqrt{2a\tau_o} = v_o$ – the velocity of the jet at very distant points, $\rho_o(1 - \tau_o)^{-\beta} = \rho_1$ the density of gas at the same points, the required formula for R is obtained:

$$R = 2l\rho_1 v_o^2 \frac{\pi}{\pi + \frac{8}{1 - \cos m}\sum_1^{\infty}(-1)^{n-1}\frac{nx_{n,0}}{4n^2 - 1}(1 - \cos 2nm)} \tag{82}$$

The angle m may be determined by equation (80) in which all magnitudes except m are given. Thus

$$v_o = \sqrt{2a\tau_o}, \quad a = k\frac{\gamma}{\gamma - 1}\rho_o^{\gamma-1} = \frac{\gamma}{\gamma - 1}\frac{p_o}{\rho_o} = \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1}\frac{1}{1 - \tau_o}$$

where p_o and ρ_o are the pressure and density at the critical point of the branching line of flow; whence

$$\tau_o = \frac{\rho_1 v_o^2}{\rho_1 v_o^2 + \frac{2\gamma}{\gamma - 1}p_1}, \quad s_o = \frac{\tau_o}{1 - \tau_o} = \frac{(\gamma - 1)\rho_1 v_o^2}{2\gamma p_1} \tag{82'}$$

Finally, the difference between the values of ψ on the jet boundaries (a magnitude will be denoted by Q) is determined from the condition

$$Q\rho_o = 2\rho_1 b v_o = 2bv_o(1 - \tau_o)^{\beta}\rho_o$$

where 2b is the width of the jet at infinity. The magnitudes b, v_o, ρ_1 and p_1 should, of course, be considered as given.

The resultant pressure, after m has been found, is, of course, most simply computed by formula (81). It may be remarked here, incidentally, that this formula may be derived very simply from the momentum theorem. It will then be found that the formula in no way depends either on the shape of the plate, provided that the latter is symmetrical with respect to the center line of flow, or on the relation between the pressure and the density. Thus, denote by M the momentum of the gas enclosed in the jet bounded on one side by the perpendicular section of the initial jet at a very large distance and on the other side by two similar sections passing through the distant points of the branched jet. The increment in M in the infinitesimal time interval Δt is evidently equal to $2b\rho_1 v_0 \Delta t(\cos m - 1)v_0 = Q\rho_0 v_0(\cos m - 1)\Delta t$, since $2b$ is the initial width of the jet and $Q\rho_0$ is the quantity of gas passing through its cross section per second. The impulse of the external forces is given by $-R\Delta t$. By the momentum theorem there is obtained

$$Q\rho_0 v_0(\cos m - 1)\Delta t = -R\Delta t$$

and this equation, after simplification, leads to formula (81). For this purpose the less general formula (82) is of more importance. By using it the second fundamental problem of the investigation may be solved approximately: namely, the pressure of a boundless gaseous fluid on an obstructing plate.

The approximate summation of the series is started by entering the denominator of formula (82). For this, the approximate expressions for the functions x_n already used may be used again in the problem of the outflow of a gas from a vessel (in deriving formula (61):

$$2x_n = 2 - 5s - \frac{25s^2}{2(2 - 5s)} + \frac{30s^2}{2 + 19s}\,\frac{1}{n + 1}$$

$$+ \frac{30s^2}{n + \mu}\left[\frac{2}{(2 - 5s)^2} - \frac{1}{2 + 19s}\right]$$

where

$$\mu = \frac{6 + 9s}{4 - 10s}, \quad s = \frac{\tau}{1 - \tau}$$

Denoting for briefness the computed Σ by L yields

$$2L = \left[2-5s-\frac{25s^2}{2(2-5s)}\right]\sum_1^\infty \frac{(-1)^{n-1}n}{4n^2-1}(1-\cos 2nm)+$$

$$+\frac{30s^2}{2+19s}\sum_1^\infty \frac{(-1)^{n-1}n}{(n+1)(4n^2-1)}(1-\cos 2nm)+ \qquad (83)$$

$$+30s^2\left[\frac{2}{(2-5s)^2}-\frac{1}{2+19s}\right]\sum_1^\infty \frac{(-1)^{n-1}n}{(n+\mu)(4n^2-1)}(1-\cos 2nm).$$

The series of the first, second, and third row will be denoted by N_1, N_2, N_3, respectively, and it will be noted that after computing N_3 and N_1, N_2 is obtained by substituting $\mu = 1$ in N_3. Then N_1 can be considered as the limiting value of the magnitude N_1';

$$N_1' = \sum_{n=1}^{n=k}(-1)^{n-1}\frac{n}{4n^2-1}(1-\cos 2nm); \quad N_1 = \lim_{k=\infty} N_1'.$$

There is obtained:

$$4N_1' = \sum_{n=1}^{n=k}(-1)^{n-1}\left(\frac{1}{2n-1}+\frac{1}{2n+1}\right)-\sum_{n=1}^{n=k}(-1)^{n-1}\frac{\cos 2nm}{2n-1}-$$

$$-\sum_{n=1}^{n=k}(-1)^{n-1}\frac{\cos 2nm}{2n+1};$$

The first of these sums has the value

$$1+(-1)^{k-1}\frac{1}{2k+1};$$

the second may be given in the form

$$\cos m\sum_1^k(-1)^{n-1}\frac{\cos(2n-1)m}{2n-1}-\sin m\sum_1^k(-1)^{n-1}\frac{\sin(2n-1)m}{2n-1},$$

and the third in the form

$$-\cos m\sum_2^{k+1}(-1)^{n-1}\frac{\cos(2n-1)m}{2n-1}-\sin m\sum_2^{k+1}(-1)^{n-1}\frac{\sin(2n-1)m}{2n-1}.$$

By the use of these formulas there is obtained for $4N_1'$ the expression

$$4N_1' = (-1)^{k-1}\frac{1-\cos 2km}{2k+1}+2\sin m\sum_1^k(-1)^{n-1}\frac{\sin(2n-1)m}{2n-1},$$

whence

$$N_1=\frac{\sin m}{2}\lim_{k=\infty}\sum_1^k(-1)^{n-1}\frac{\sin(2n-1)m}{2n-1}.$$

But

$$\sum_1^k(-1)^{n-1}\frac{\sin(2n-1)m}{2n-1}=\int_0^m\sum_1^k(-1)^{n-1}\cos(2n-1)m\,.\,dm,$$

$$\sum_1^k(-1)^{n-1}\cos(2n-1)m=\frac{1}{2\cos m}\sum_1^k(-1)^{n-1}(\cos 2nm+$$

$$+\cos(2n-2)m)=\frac{1}{2\cos m}[1+(-1)^{k-1}\cos 2km],$$

hence

$$N_1=\frac{\sin m}{4}\int_0^m\frac{dm}{\cos m}+\lim_{k=\infty}(-1)^{k-1}\frac{\sin m}{4}\int_0^m\frac{\cos 2km}{\cos m}\,dm.$$

The last term is zero. For

$$\int_0^m\frac{\cos 2km}{\cos m}\,dm=\int_0^m\frac{\cos(2k-1)m\cos m-\sin(2k-1)m\sin m}{\cos m}\,dm=$$

$$=\frac{\sin(2k-1)m}{2k-1}-\int_0^m\frac{m\sin m}{\cos m}\;\frac{\sin(2k-1)m}{m}\,dm;$$

and from the properties of the Fourier integrals

$$\lim_{k=\infty}\int_0^m\frac{m\sin m}{\cos m}\;\frac{\sin(2k-1)m}{m}\,dm=0\ {}^{*)}.$$

Hence

$$\lim_{k=\infty}\int_0^m\frac{\cos 2km}{\cos m}\,dm=0.$$

Thus

$$N_1=\sum_1^\infty(-1)^{n-1}\frac{n}{4n^2-1}(1-\cos 2nm)=\frac{\sin m}{4}\int_0^m\frac{dm}{\cos m}=$$

$$=\frac{\sin m}{4}\log cotg\left(\frac{\pi}{4}-\frac{m}{2}\right) \tag{84}$$

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*See reference 7, p. 233.

Now turn to the computation of N_3.

$$N_3=\sum_1^\infty \frac{(-1)^{n-1}n}{(n+\mu)(4n^2-1)}(1-\cos 2nm)=$$

$$=\int_0^1\sum_1^\infty(-1)^{n-1}\frac{nt^{n+\mu-1}}{4n^2-1}(1-\cos 2nm)dt.$$

The function under the integral may be expressed in finite form through the lower transcendentals. For this purpose, consider it as the real part of

$$\sum_1^\infty(-1)^{n-1}\frac{nt^{n+\mu-1}}{4n^2-1}(1-e^{2nmi}).$$

The integral N_3 may then be expressed as

$$R\int_0^1\sum_1^\infty(-1)^{n-1}\frac{nt^{n+\mu-1}}{4n^2-1}(1-e^{2nmi})dt;$$

where R is used to indicate the real part of the complex quantity following it. But

$$4\sum(-1)^{n-1}\frac{nt^n}{4n^2-1}=\Big[t+\frac{t}{3}-\frac{t^2}{3}-\frac{t^2}{5}+\frac{t^3}{5}+\frac{t^3}{7}-\frac{t^4}{7}-$$

$$-\frac{t^4}{9}+\ldots\Big]=\sqrt{t}\,arctg\sqrt{t}-\frac{1}{\sqrt{t}}(arctg\sqrt{t}-\sqrt{t})=$$

$$=1-\frac{1-t}{\sqrt{t}}\,arctg\sqrt{t};$$

$$4\sum(-1)^{n-1}\frac{nt^ne^{2nmi}}{4n^2-1}=1-\frac{1-te^{2mi}}{\sqrt{t}e^{mi}}\,arctg\sqrt{t}e^{mi}.$$

Thus

$$4N_3=R\int_0^1 t^{\mu-\frac{3}{2}}\{(e^{-mi}-te^{mi})arctg\sqrt{t}e^{mi}-(1-t)arctg\sqrt{t}\}dt.$$

By applying integration by parts, there is obtained

$$J_m=\int_0^1 t^{\mu-\frac{3}{2}}(e^{-mi}-te^{mi})arctg\sqrt{t}e^{mi}dt=$$

$$=\left(\frac{e^{-mi}}{\mu-\frac{1}{2}}-\frac{e^{mi}}{\mu+\frac{1}{2}}\right)arctg\,e^{mi}+\frac{1}{\mu(2\mu+1)}-\frac{4\mu}{4\mu^2-1}\int_0^1\frac{t^{\mu-1}}{1+te^{2mi}}dt;$$

but

$$\frac{e^{-mi}}{\mu-\frac{1}{2}}-\frac{e^{mi}}{\mu+\frac{1}{2}}=\frac{4\cos m}{4\mu^2-1}-\frac{8\mu i\sin m}{4\mu^2-1};$$

$$arctge^{im}=\frac{\pi}{4}+\frac{i}{2}\lg cotg\left(\frac{\pi}{4}-\frac{m}{2}\right);$$

$$\int_0^1\frac{t^{\mu-1}}{1+te^{2mi}}dt=\int_0^1\frac{(1+t\cos 2m)t^{\mu-1}}{1+2t\cos 2m+t^2}dt-i\sin 2m\int_0^1\frac{t^{\mu}dt}{1+2t\cos 2m+t^2};$$

thus

$$J_m=\frac{\cos m-2\mu i\sin m}{4\mu^2-1}\left[\pi+2i\lg\cot g\left(\frac{\pi}{4}-\frac{m}{2}\right)\right]-$$

$$-\frac{4\mu}{4\mu^2-1}\int_0^1\frac{(1+t\cos 2m)t^{\mu-1}}{1+2t\cos m+t^2}dt+\frac{4\mu i\sin 2m}{4\mu^2-1}\int_0^1\frac{t^{\mu}dt}{1+2t\cos m+t^2}.$$

The integral in the formula for N_3 is equal to J_m for $m = 0$. Finally, there is obtained

$$4N_3=\frac{\pi(\cos m-1)}{4\mu^2-1}+\frac{4\mu\sin m}{4\mu^2-1}\lg cotg\left(\frac{\pi}{4}-\frac{m}{2}\right)+\frac{4\mu}{4\mu^2-1}\int_0^1\frac{t^{\mu-1}dt}{1+t}-$$

$$-\frac{4\mu}{4\mu^2-1}\int_0^1\frac{(1+t\cos 2m)t^{\mu-1}}{1+2t\cos 2m+t^2}dt;$$

$$\frac{4N_3}{1-\cos m}=-\frac{\pi}{4\mu^2-1}+\frac{4\mu}{4\mu^2-1}\,\frac{\sin m}{1-\cos m}\lg cotg\left(\frac{\pi}{4}-\frac{m}{2}\right)-$$

$$-\frac{8\mu(1+\cos m)}{4\mu^2-1}\int_0^1\frac{t^{\mu}(1-t)dt}{(1+t)(1+t^2+2t\cos 2m)}. \tag{85}$$

Setting in this formula $\mu = 1$ yields finally the value $4N_2$ (N_2 is the second summation entering the formula (83)).

$$\frac{4N_2}{1-\cos m}=-\frac{\pi}{3}+\frac{4}{3}\,\frac{\sin m}{1-\cos m}\lg cotg\left(\frac{\pi}{4}-\frac{m}{2}\right)-$$

$$-\frac{8(1+\cos m)}{3}\int_0^1\frac{t(1-t)dt}{(1+t)(1+t^2+2t\cos 2m)}; \tag{86}$$

The definite integral in the above equation is easily computed; it is equal to

$$m\cot g m+\frac{\cos 2m}{1-\cos 2m}\lg\cos m-\lg 2. \tag{87}$$

Everything required for the computation of 2L by formula (83) has been developed and the expression for the force on a plate for any gas jet can be set up. Now, consider the case where the jet is infinitely wide – that is, the problem of the action of a boundless gas stream on a plate. For this condition $m = 0$, since the flow after passing round the plate must finally resume its initial direction.

Therefore compute

$$\lim_{m=0} \frac{8L}{1 - \cos m}$$

where the exact value of L is given by the series entering the denominator of formula (82) and the approximate value by relation (83). Using formulas (84), (85), (86), and (87), there is obtained

$$\lim_{m=0} \frac{N_1}{1 - \cos m} = \frac{1}{2}, \quad \lim_{m=0} \frac{N_2}{1 - \cos m} = \frac{4}{3}\lg^2 - \frac{4 + \pi}{12} = 0.32907$$

$$\lim_{m=0} \frac{N_3}{1 - \cos m} = \frac{2\mu}{4\mu^2 - 1} - \frac{\pi}{4(4\mu^2 - 1)} - \frac{4\mu}{4\mu^2 - 1}\int_0^1 \frac{t^\mu (1 - t)dt}{(1 + t)^3}$$

and since

$$\int_0^1 \frac{t^\mu (1 - t)}{(1 + t)^3}dt = \frac{2\mu + 1}{4} - \mu^2 \int_0^1 \frac{t^{\mu-1}dt}{1 + t}$$

therefore

$$\lim_{m=0} \frac{N_3}{1 - \cos m} = \frac{4\mu^3}{4\mu^2 - 1}\int_0^1 \frac{t^{\mu-1}dt}{1 + t} - \frac{\mu}{2\mu + 1} - \frac{\pi}{4(4\mu^2 - 1)}$$

Next, by (83)

$$\lim_{m=0} \frac{8L}{1 - \cos m} = 4 - 10s - \frac{25s^2}{2 - 5s} + \frac{30s^2}{2 + 19s} 1.3163$$

$$+ 30s^2\left[\frac{2}{(2 - 5s)^2} - \frac{1}{2 + 19s}\right]\left(\frac{16\mu^3}{4\mu^2 - 1}\int_0^1 \frac{t^{\mu-1}dt}{1 + t}\right.$$

$$\left. - \frac{4\mu}{2\mu + 1} - \frac{\pi}{4\mu^2 - 1}\right) = P(s)$$

where

$$\mu = \frac{6 + 9s}{4 - 10s}, \quad s = \frac{\tau}{1 - \tau}, \quad s = s_0 \tag{88'}$$

Formula (82) takes the form

$$R = 2l\rho_1 v_0^2 \frac{\pi}{\pi + P} \tag{88}$$

s_0 being determined by formula (82'):

$$s_0 = \frac{(\gamma - 1)\rho_1 v_0^2}{2\gamma p_1}$$

which may be rewritten as

$$s_0 = \frac{v_0^2}{5c^2}$$

by substituting the velocity of sound c at the distant points of the flow and the value 1.40 for γ.

For $s = 0$ the expression found for R gives the formula of Kirchhoff (in reference 2):

$$R = 2l\rho_1 v_0^2 \frac{\pi}{\pi + 4} = 0.44 \times 2l\rho_1 v_0^2$$

applicable for the pressure of the flow of an incompressible liquid. This value is approached by the accurate formula (82) for $s = 0$. The greater the velocity, however, the greater the magnitude $\frac{\pi}{\pi + P}$, and therefore the reaction of the gas flow with increasing velocity increases somewhat more energetically than in the case of liquid flow. Now, compute the coefficient $\frac{\pi}{\pi + P}$ for the limiting value of s equal to 0.2 and for values near the limit, to an accuracy of 0.01.

Setting $s_0 = 0.2$ yields $\mu = 3.9$, $\int_0^1 \frac{t^{\mu-1}}{1+t}dt = j(\mu) = 0{,}144$;

$$k(\mu) = \frac{16\mu^3}{4\mu^2-1}\,j(\mu) - \frac{4\mu}{2\mu+1} - \frac{\pi}{4\mu^2-1} = 0{,}459;$$

$$P(0{,}2) = 1 + \frac{6}{29}\,0{,}857 + \frac{12}{5}\cdot 0{,}459 = 2{,}28.$$

$$\frac{\pi}{\pi+P} = 0{,}58;$$

$v_0 = c$, the velocity of sound at the distant points of the gas jet; if p_1, the pressure in that region is equal to 1 atmosphere; then in the case of air $c = 333$ meters per second.

For $s_0 = \frac{2}{11} = 0{,}1818$; $\mu = 3{,}5$; $v_0 = c\sqrt{\frac{10}{11}} =$ about $318^m/_{sec}$;

$$j(\mu) = 0{,}1625;\quad k(\mu) = 0{,}508;$$

$$P(0{,}1818) = 4 - 1{,}818 - \frac{1{,}818^2}{4.1{,}091} + \frac{3.1{,}818^2}{54{,}54}\,0{,}808 +$$

$$+ \frac{3.1{,}818^2}{1{,}091^2}\,0{,}102 = 2{,}42;\quad \frac{\pi}{\pi+P} = 0{,}56.$$

For $s_0 = \frac{2}{17} = 0{,}1176$, $\mu = 2{,}5$; $v_0 = \sqrt{\frac{10}{17}}\,c = 255^m/_{sec}$.

$$j(\mu) = 0{,}2375;\quad k(\mu) = 0{,}676.$$

$$P(0{,}1176) = 4 - 1{,}176 - \frac{1{,}176^2}{4.1{,}412} + \frac{3.1{,}176^2}{42{,}354}\,.0{,}640 +$$

$$+ \frac{3.1{,}176^2}{1{,}412^2}\cdot 0{,}135 = 2{,}93;\quad \frac{\pi}{\pi+P} = 0{,}52.$$

For $s_0 = \frac{2}{29} = 0{,}069$, $\mu = 2$; $v_0 = \sqrt{\frac{10}{29}}\,c = 195{,}5^m/_{sec}$.

$$j(\mu) = 0{,}3069,\quad k(\mu) = 0{,}8085;$$

$$P(0{,}069) = 4 - 0{,}69 - \frac{0{,}69^2}{4.1{,}655} + \frac{3.0{,}69^2}{33{,}11}\cdot 0{,}508 +$$

$$+ \frac{3.0{,}69^2}{1{,}655^2}\,0{,}162 = 3{,}34;\quad \frac{\pi}{\pi+P} = 0{,}485.$$

For $s_0 = \frac{2}{53} = 0{,}0377$; $\mu = 1{,}75$; $v_0 = \sqrt{\frac{10}{53}}\,c = 144{,}6^m/_{sec}$;

$$j(\mu) = 0{,}3578;\quad k(\mu) = 0{,}8925;$$

$$P(0{,}0377) = 4 - 0{,}377 + 0{,}010 = 3{,}63;$$

$$\frac{\pi}{\pi+P} = 0{,}464.$$

It is thus seen that the coefficient of the formula for the pressure drops sharply with decrease in velocity near the limiting velocities; after which it drops more slowly. Thus, when the velocity decreases from 333 meters per second to 318 meters per second, the coefficient drops from 0.58 to 0.56 - decreasing by 0.02 - that is, 3.4 percent of its value. The same numerical value for the drop is obtained on changing the velocity from 196 meters per second to 145 meters per second, although the difference of these velocities is 3.4 times greater. It may be noted, moreover, that at a velocity of 145 meters per second the pressure coefficient is already near the value which is computed by the formula of Kirchhoff $\frac{\pi}{\pi + 4} = 0.44$; the differences of these values is equal to 0.024, about 5 percent of the greater of them. The total increment of the coefficient 0.58 - 0.44 = 0.14 is about 32 percent of its lower value.

Thus, at not too large velocities the coefficient in the formula for the pressure on a plate, or what is equivalent, the resistance of a gaseous medium to the motion in it of a plate increases very slowly. Therefore under these conditions the resistance of the medium follows approximately the square law. When the velocity of motion of the plate is near the velocity of sound, however, the resistance increases in a very marked manner. This conclusion is entirely confirmed by the available experimental data, as is shown later.

Further is noted a relatively simple formula which for the assumed accuracy of computation gives results entirely agreeing with those obtained by formula (88):

$$P(s_0) = 4 - 10s_0 + 7s_0^2 \tag{89}$$

Compute $P(s_0)$ by (89) and by (88); and for comparison write the results one below the other. The following table is obtained

s_0	0.2	0.1818	0.1176	0.069	0.0377
$P(s_0)$ by (88)	2.28	2.42	2.93	3.34	3.63
$P(s_0)$ by (89)	2.28	2.41	2.92	3.34	3.63

The difference obtained in the value of P by these formulas in no way affects the value of the resistance coefficient for the assumed approximation.

The final resistance equation now assumes the form

$$R = \frac{\pi}{\pi + 4 - 10s_o + 7s_o^2} 2l\rho_1 v_o^2 \qquad (90)$$

where

$$s_o = \frac{v_o^2}{5c^2}$$

where c is the velocity of sound propagation at the distant points of the flow, equal to 333 meters per second if the motion of the plate takes place in the atmosphere.

If v_o is not very large $\left(v_o < \frac{c}{\sqrt{10}}\right)$, the term $7s_o^2$ is negligible within the limits of accuracy; then

$$R = \frac{\pi}{\pi + 4 - 2\frac{v_o^2}{c^2}} 2l\rho_1 v_o^2 \qquad (90')$$

The approximate formula (90) may be obtained from the exact expression for the pressure of the gas jet on the plate in exactly the same manner as the corresponding formula (69) in the previous section. It is not difficult to show that if in the denominator of relation (87) $x_{n,o}$ is replaced approximately by

$$x_n(0) + s_o x'_n(0) + \frac{s_o^2}{2} x_o''(0)$$

the computation is carried out and m is set equal to 0 formula (90) is arrived at. The difference will be only that the denominator will be found equal to $\pi+4-10s_o+7.2s_o^2$, but this is of small significance for assumed accuracy of computation.

Investigation will now be made to find what formula (90) will yield if it is attempted to apply it in determining air resistance to motion, using it beyond the true limit of its applicability.

The fundamental factor to be considered is the coefficient K

$$K = \frac{\pi}{\pi + 4 - 10s_o + 7s_o^2}$$

As long as the velocity of the plate does not exceed the velocity of sound this coefficient increases at first very slowly, then much more rapidly with increase in velocity, as has been shown, the limits of its variations being given by the extreme values 0.44 and 0.58. This increase continues even after s_o goes beyond the value 0.2 corresponding to the equation $v_o = c$; thus for $v_o = \frac{3}{2}c$ = about 500 meters per second, $s_o = 0.45$, $K = 0.77$; for $s_o = \frac{5}{7}$, $v_o = \frac{5}{7}\sqrt{7c}$ = about 629 meters per second; K attains its maximum value 0.88, twice its value for small velocities. Further on K decreases and for $v_o = 2.5c$ = about 833 meters per second, $s_o = 1.25$, there is obtained $K = 0.56$.

The above results qualitatively are in sufficiently good agreement with test results. This is all the more interesting in view of the fact that the tests were conducted under conditions very far removed from those of the theoretical problem considered since, in fact, K was computed from observations on the flight of artillery projectiles. The results were obtained from ballistic tests by Zabudsky (reference 11, pp. 47-57, table 4, and fig. 30) and were mainly used for the purposes of comparison. The change in the coefficient K for velocities not exceeding 240 meters per second is in fact almost inappreciable; it then starts to grow very rapidly, increasing 2.8 times for a change in the velocity of the projectile from 240 to 420 meters per second; thereafter it remains at the same level until the velocity exceeds 550 meters per second and then drops, giving for 1100 meters per second the same value as for 340 meters per second. Thus the actual change in K stands out with great sharpness: the law of the proportionality of the resistance to the square of the velocity clearly applies for velocities not too large, while for velocities near that of sound the coefficient increases much more rapidly. This phenomenon might to a certain degree have been foreseen. Thus in the theoretical problem the tubes of flow separate from the moving plate in only two directions, while in the flight of an artillery shell they separate in all meridional planes. If in each of these planes the motion took place as in the case under consideration the resistance coefficient should vary as K^2. Actually the deviations are not so large. This is because the tubes of flow springing from the projectile spread out and therefore should press less strongly on the body of the projectile near its contours. Better results could not be expected for the reason that applied formula (90) is outside the limits within which its applicability has been proved.

Aside from the above reasons there is yet a further deviating factor; namely, the viscosity of the air and its friction at the side of the moving body. Owing to the viscosity there should be formed behind the plate vortices which lower the pressure in this region; and hence lead to an increase in the resistance. This condition already shows up at the smaller velocities such that, as shown by the tests of Tibot, the coefficient K is equal approximately to 0.64 instead of 0.44 if the velocity fluctuates within the limits of 0.5 to 11 meters per second (reference 11, p. 14). For large velocities the effect of the viscosity would presumably be not so large.

In concluding this part, a method is indicated for deriving a theoretical formula for the resistance in the case where the velocity of motion of the plate exceeds the velocity of sound. In this case, for the same reasons as for the case of a gas flowing out of a vessel, no continuous steady motion should be expected. As in the previous case, a certain partition surface should be formed dividing the region of the flow into two parts in each of which the motion possesses a different character. This surface, consisting of the two sheets shown in figure 6, will be considered as enveloping the sound waves. Within the compressed and heated air layer separated by the surface from the atmosphere the motion will be steady and the variable τ will everywhere be less than $\frac{1}{2\beta + 1}$; on the surface itself $\tau = \frac{1}{2\beta + 1}$ and the relative velocity of the air particles penetrating the layer is equal to the sound velocity which would be observed at this place in a gas at rest and which is equal to the velocity of the moving plate. On passing through the boundary of the layer into the outer atmosphere, a sharp drop in pressure is encountered; here the motion will be unsteady. Under the same assumption as in the case of the outflow from a vessel it is found that the angle at which the flow tubes in their relative motion (for stationary plate) intersect the boundary of the region of the condensed layer will be constant at all points of the partition surface. This additional condition is sufficient for a mathematical analysis of the motion within the separating air layer and, therefore, also for the solution of the problem of the air pressure on the plate. It may be remarked that the very existence of the partition surface and condensed air layer are by no means to be considered as hypothetical, since the existence of these phenomena has been firmly established by Mach and other careful investigators.

PART V

APPROXIMATION METHOD OF SOLUTION OF GAS JET PROBLEMS

If the velocities of the gas flow are sufficiently below the limiting velocity c determined by the equation $\tau_o = \frac{1}{2\beta+1}$ the solution formulas may be presented, approximately, in a more simple and compact form by introducing a certain complex variable.

In part I the following fundamental equations connecting the derivatives of the velocity potential and stream function with respect to the independent variables τ and θ were derived:

$$\frac{\partial\varphi}{\partial\theta} = 2\tau(1-\tau)^{-\beta}\frac{\partial\psi}{\partial\tau}$$

$$\frac{\partial\varphi}{\partial\tau} = -\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)}(1-\tau)^{-\beta}\frac{\partial\psi}{\partial\theta}$$

These are formulas (11) of part I. The following notation is introduced:

$$\int_{\tau}^{\tau_o} \frac{(1-\tau)^{\beta}}{2\tau}\,d\tau = \sigma \qquad (91)$$

where τ_o is the maximum value of τ corresponding to the boundary of the jet. The preceding formulas then can be expressed by

$$\frac{\partial\varphi}{\partial\theta} = -\frac{\partial\psi}{\partial\sigma}$$

$$\frac{\partial\varphi}{\partial\sigma} = K\frac{\partial\psi}{\partial\theta}$$

where

$$K = \frac{1-(2\beta+1)\tau}{(1-\tau)^{2\beta+1}}$$

In the region of flow the coefficient K varies; but <u>if the velocities of motion of the gas are not near the limiting velocity K is confined within very narrow limits</u>.

First it is shown that K decreases with increasing τ. For this purpose the derivative $dK/d\tau$ is obtained:

$$\frac{dK}{d\tau} = -\frac{2\beta(2\beta+1)\tau}{(1-\tau)^{2\beta+2}}$$

It is clear that the minus sign is retained, whatever the positive value τ, so that the foregoing statement is correct. Next the values of K for the extreme values of τ admissible in the problem considered is computed. Then on the basis of previously mentioned data K will be included between the boundary values thus obtained.

It is necessary to proceed, for convenience of the computation, from the variable τ to the variable $s = \tau/(1-\tau)$ to obtain, for K, the value

$$K = (1-2\beta s)\ (1+s)^{2\beta}$$

or if β is set, as before, equal to 2.5

$$K = (1-5s)\ (1+s)^{5}$$

whence is obtained

$$s_o = \frac{1}{12\beta} = \frac{1}{30}, \qquad 1 > K > 0.982;$$

$$s_o = \frac{1}{14\beta} = \frac{1}{35}, \qquad 1 > K > 0.987$$

$$s_o = \frac{1}{16\beta} = \frac{1}{40}, \qquad 1 > K > 0.990$$

$$s_o = \frac{1}{18\beta} = \frac{1}{45}, \qquad 1 > K > 0.9920$$

$$s_o = \frac{1}{24\beta} = \frac{1}{60} \qquad 1 > K > 0.9957$$

$$s_o = \frac{1}{32\beta} = \frac{1}{80} \qquad 1 > K > 0.9976$$

$$s_o = \frac{1}{50\beta} = \frac{1}{125}, \qquad 1 > K > 0.9992$$

$$s_o = \frac{1}{200\beta} = \frac{1}{500}, \qquad 1 > K > 0.99995$$

The corresponding values of the maximum velocity v_o are determined by the formula $v_o^2 = 5c^2 s_o$, where c is the velocity of sound for the physical state of the gas at the boundaries of the jet. For the preceding values of s_o, if it is assumed that near the boundaries of the jet mean atmospheric conditions prevail, the following values are obtained for v_o:

136; 126; 118; 111; 96; 83; 66.6; 33.3 meters per

second (the figures are rounded for simplicity; c is assumed equal to 333 m/sec).

The approximation which is now made consists in taking K equal to unity. Then there is obtained:

$$\frac{\partial\varphi}{\partial\sigma} = \frac{\partial\psi}{\partial\theta}$$

$$\frac{\partial\varphi}{\partial\theta} = -\frac{\partial\psi}{\partial\sigma}$$

and, therefore

$$w = \varphi + i\psi = F(\sigma + i\theta) \tag{92}$$

where σ is determined by formula (91).

It is well to consider again the corresponding motion of an incompressible liquid for the same boundary conditions, together with the problem on the gas flow; plane bounding walls, flow extends infinitely in certain directions, behind the walls at which the flowing mass separates the pressure is constant and the liquid or the gas is at rest. The problem for the case of the incompressible liquid is solved by the relations:

$$w_1 = \varphi_1 + i\psi_1 = F(\vartheta + i\theta), \quad \vartheta + i\theta = \lg \frac{v_0 dz}{dw_1}$$

where $\vartheta = \lg \frac{v_0}{v}$, θ is the same angle of the velocity with the X axis as in the gas-flow problem. Over all the boundaries ψ_1 has some constant value; at the at the bounding walls θ = constant; and at the jet surface the velocity v = constant = v_0; and therefore $\vartheta = 0$. These are the conditions imposed on w_1, a function of the complex variable $\vartheta + i\theta$. The method of obtaining such function is given by Joukowsky in reference 2.

It is clear that after the function F is found, which solves the given problem on the incompressible liquid, the required solution of the same problem on the gas motion is obtained by setting

$$\varphi + i\psi = F(\sigma + i\theta)$$

that is, simply replacing ϑ by σ; then when the variables θ and σ pass around the boundaries of their region, ψ will receive the same constant values as ψ_1; where $\vartheta = 0$ of course $\sigma = 0$, and, therefore, $\tau = \tau_0$.

After the function $\varphi + i\psi$ is found as a function of $\sigma + i\theta$ the coordinates easily can be found as functions of the variables σ and θ, the contours of the jet investigated can be obtained and the constants characteristic of the problem determined: namely, the quantity of gas carried by the jet and the resultant force on the plates.

To determine the dependence of the coordinates on σ and θ τ is expressed in terms of σ. From formula (91)

$$\frac{d}{d\sigma}\frac{1}{\sqrt{\tau}}=-\frac{1}{2\tau\sqrt{\tau}}\frac{d\tau}{d\sigma}=\frac{(1-\tau)^{-\beta}}{\sqrt{\tau}},$$

$$\frac{d}{d\sigma}\frac{(1-\tau)^{-\beta}}{\sqrt{\tau}}=-\frac{1-(2\beta+1)\tau}{2\tau\sqrt{\tau}}(1-\tau)^{-\beta-1}\frac{d\tau}{d\sigma}=\frac{K}{\sqrt{\tau}};$$

and since it is assumed that

$$K=[1-(2\beta+1)\tau]:(1-\tau)^{2\beta+1}=1,$$

there is obtained

$$\frac{d^2}{d\sigma^2}\frac{1}{\sqrt{\tau}}=\frac{1}{\sqrt{\tau}};\ \frac{1}{\sqrt{\tau}}=\frac{C_1e^{\sigma}+C_2e^{-\sigma}}{2};\ \frac{(1-\tau)^{-\beta}}{\sqrt{\tau}}=\frac{C_1e^{\sigma}-C_2e^{-\sigma}}{2}. \quad (93)$$

Setting $\tau = \tau_0$ yields

$$\frac{C_1+C_2}{2}=\frac{1}{\sqrt{\tau_0}},\ \frac{C_1-C_2}{2}=\frac{(1-\tau_0)^{-\beta}}{\sqrt{\tau_0}}. \quad (93')$$

Turn now to formulas (7') and (8), part I, that gives the derivatives of the coordinates with respect to φ and ψ:

$$\frac{\partial x}{\partial\varphi}=\frac{\cos\theta}{\sqrt{2\alpha\tau}},\quad \frac{\partial y}{\partial\psi}=\frac{\sin\theta}{\sqrt{2\tau\alpha}};$$

$$\frac{\partial x}{\partial\psi}=-\frac{\sin\theta(1-\tau)^{-\beta}}{\sqrt{2\alpha\tau}},\quad \frac{\partial y}{\partial\psi}=\frac{\cos\theta(1-\tau)^{-\beta}}{\sqrt{2\tau\alpha}}.$$

whence, by taking into account (93) and setting $x + iy = z$, $\varphi + i\psi = w$, $\varphi - i\psi = w'$ there is obtained:

$$2\sqrt{2\alpha}\frac{\partial z}{\partial\varphi}=C_1e^{\sigma+i\theta}+C_2e^{-\sigma+i\theta};\quad 2\sqrt{2\alpha}\frac{\partial z}{\partial\psi}=i(C_1e^{\sigma+i\theta}-C_2e^{-\sigma+i\theta});$$

$$2\sqrt{2\alpha}\frac{\partial z}{\partial\sigma}=C_1e^{\sigma+i\theta}\frac{\partial w}{\partial\sigma}+C_2e^{-\sigma+i\theta}\frac{\partial w'}{\partial\sigma}; \quad (94)$$

$$2\sqrt{2\alpha}\frac{\partial z}{\partial\theta}=C_1e^{\sigma+i\theta}\frac{\partial w}{\partial\theta}+C_2e^{-\sigma+i\theta}\frac{\partial w'}{\partial\theta};$$

$$2\sqrt{2\alpha}\,z=C_1\int e^{\sigma+i\theta}dw+C_2\int e^{-\sigma+i\theta}dw'.$$

The integration now may be carried out, since

$$w=f(\sigma+i\theta),\ w'=f_1(\sigma-i\theta),$$

where f_1 is the function conjugate to f.

Consider, for example, the approximate solution of the problem of the pressure exerted on a plate by an infinite gas flow, or otherwise expressed, of the resistance of a gas medium to the motion of a plate. It is assumed here that the direction of the flow forms a certain angle λ with the normal to the plate. By making use of the Joukowsky method the following solution of this problem for the incompressible liquid is readily obtained:

$$\varphi_1 + i\psi_1 = ku^2$$

$$\frac{\cos^2\lambda}{u} = \sin\lambda + \sin(\theta - i\vartheta)$$

The regions of flow correspond, in this case, to the upper half plane of the variable u. The boundaries $\vartheta = 0$, which determine the streamlines CA and BD, correspond to the segments of the real axis of the u region from $u = +\infty$ to $u = 1 - \sin\lambda$ and from $u = -1 - \sin\lambda$ to $u = -\infty$ the point $u = \pm\infty$ gives $\vartheta = 0$, $\theta = -\lambda$. The part of the plate where $\theta = \pi/2$ corresponds to the segment of the real axis of the u plane bounded by the points $u = 0$ and $u = 1 - \sin\lambda$; finally, at the boundary OB $\theta = -\pi/2$ and u varies from 0 to $-1 - \sin\lambda$.

On the other hand, for u real $\psi_1 = 0$ and φ_1 varies from 0 to $+\infty$ as u varies from 0 to $\pm\infty$. The imaginary axis of the u region likewise gives $\psi_1 = 0$; φ_1 increases from $-\infty$ to 0, while u runs through the values from $+\infty i$ to 0; ϑ and θ vary correspondingly within the limits 0 and $+\infty$, $-\lambda$ and 0. It is clear, from this, that the imaginary axis of the u half plane corresponds in the plane of flow to the streamline, EO branching at the plate into OAC and OBD.

On the basis of the foregoing rule, the solution of this problem of the gas flow is obtained by setting

$$\varphi + i\psi = ku^2, \quad \frac{\cos^2\lambda}{u} = \sin\lambda + \sin(\theta - i\sigma) \tag{95}$$

The expressions for the coordinates in terms of σ and θ now will be sought. Turning for this purpose to the last of formulas (94)

$$2\sqrt{2}\alpha z = C_1 e^{\sigma+i\theta} w + C_2 e^{-\sigma+i\theta} w' - C_1 \int w e^{\sigma+i\theta} d(\sigma+i\theta) + \\ + C_2 \int w' e^{-\sigma+i\theta} d(\sigma-i\theta).$$

Setting, for briefness, under the integral signs

$$e^{\sigma+i\theta} = t, \quad e^{-\sigma+i\theta} = t',$$

and substituting the expressions for w and w' from (95) gives

$$2\sqrt{2}\alpha z = C_1 e^{\sigma+i\theta} w + C_2 e^{-\sigma+i\theta} w' - 4kC_1 \cos^4\lambda \int \frac{t^2 dt}{(2\sin\lambda t + i - it^2)^2} - \\ - 4kC_2 \cos^4\lambda \int \frac{t'^2 dt'}{(2\sin\lambda t' + i - it'^2)^2}.$$

Integration yields

$$2\sqrt{2}\alpha z = C_1 e^{\sigma+i\theta} w + C_2 {}^{-\sigma+i\theta} w' + 2k\cos^2\lambda C_1 \frac{it\cos 2\lambda + \sin\lambda}{2\sin\lambda t + i - it^2} + \\ + 2k\cos^2\lambda C_2 \frac{it'\cos 2\lambda + \sin\lambda}{2\sin\lambda t' + i - it'^2} - 2ik\cos\lambda C_1 \operatorname{arctg} \frac{t + i\sin\lambda}{i\cos\lambda} - \\ - 2ik\cos\lambda C_2 \operatorname{arctg} \frac{t' + i\sin\lambda}{i\cos\lambda} + L.$$

Or by substitution of the sum and difference of the arctangents in the foregoing equation and multiplication of the entire equation by i:

$$\frac{2\sqrt{2}\alpha z i}{k} = C_1 i e^{\sigma+i\theta} \frac{\cos^4\lambda}{[\sin\lambda + \sin(\theta - i\sigma)]^2} + C_2 i e^{-\sigma+i\theta} \frac{\cos^4\lambda}{[\sin\lambda + \sin(\theta + i\sigma)]^2} - \\ - C_1 \cos^2\lambda \frac{\cos 2\lambda - i\sin\lambda e^{-\sigma-i\theta}}{\sin\lambda + \sin(\theta - i\sigma)} - C_2 \cos^2\lambda \frac{\cos 2\lambda - i\sin\lambda e^{\sigma-i\theta}}{\sin\lambda + \sin(\theta + i\sigma)} + \\ + \cos\lambda (C_1 + C_2) \operatorname{arctg} \left\{ \frac{(e^\sigma + e^{-\sigma}) e^{i\theta} + 2i\sin\lambda}{\cos 2\lambda + e^{2\theta i} + i\sin\lambda (e^\sigma + e^{-\sigma}) e^{\theta i}} \frac{\cos\lambda}{i} \right\} + \\ + \cos\lambda (C_1 - C_2) \operatorname{arctg} \left\{ \frac{(e^\sigma - e^{-\sigma}) e^{i\theta}}{1 - e^{2\theta i} - i\sin\lambda (e^\sigma + e^{-\sigma}) e^{\theta i}} \frac{\cos\lambda}{i} \right\} + Li. \tag{96}$$

It is not difficult to show that both arctangents entering the foregoing equation everywhere vary continuously; that as σ approaches ∞, whatever the value of θ. The first of them approaches $-\frac{\pi}{2} + \lambda$ and the second $\frac{\pi}{2} - \lambda$, and therefore, L (the arbitrary constant of integration) may be determined so that for $\sigma = \infty$, $z = 0$.

It is noted further that the first of these arctangents nowhere attains the value $\pi/2$, since the denominator of its argument nowhere becomes zero; the second passes through $\pi/2$ on the curve defined by the equation

$$2\sin\theta + \sin\lambda(e^{\sigma}+e^{-\sigma}) = 0$$

The expression for the length of the plate $2l$ is now set up. For this purpose, by formula (96) there is determined $2l = -iz_1 + iz_2$ equal to the difference in the results of the substitution in the expression $-iz$ of the values $\sigma = 0$, $\theta = \pi/2$ and $\sigma = 0$, $\theta = -\pi/2$. It may be noted that on the basis of what has been said of the variation of the second arctangent of formula (96)

$$\Bigg/_{\sigma=0,\ \theta=-\frac{\pi}{2}}^{\sigma=0,\ \theta=\frac{\pi}{2}} \operatorname{arctg} \frac{(e^{\sigma}-e^{-\sigma})e^{i\theta}\cos\lambda}{1-ie^{2\theta i}+\sin\lambda(e^{\sigma}+e^{-\sigma})e^{\theta i}} = -\pi$$

The same substitution in the first of the arctangents gives zero as a result. With this in mind, it is found, after simple reduction that:

$$\frac{2l\sqrt{2\alpha}}{k} = 4, \qquad \frac{C_1 + C_2}{2} + \pi\cos\lambda\,\frac{C_1 - C_2}{2}$$

whence, from formulas (93')

$$2l\,\frac{\sqrt{2\alpha\tau_0}}{k} = 4 + \pi\cos\lambda(1-\tau_0)^{-\beta}$$

Turn now to the computation of the resultant force R on the plate. For this purpose use is made of the formula

$$p = p_o(1-\tau)^{\beta+1}$$

where p_o is the pressure at the critical point; it is determined in terms of p_1 the pressure behind the plate and prevailing over the entire gas medium at rest by the formula

$$p_o(1-\tau_o)^{\beta+1} = p_1$$

For determining R

$$R = \int_0^{l_1} p_o(1-\tau)^{\beta+1} dy_{\theta=\frac{\pi}{2}} + \int_{-l_2}^{0} p_o(1-\tau)^{\beta+1} dy_{\theta=-\frac{\pi}{2}} - 2p_1 l$$

where l_1 and l_2 denote the corresponding lengths of the parts of the plate OA and OB from the critical point to the ends.

By carrying out the integration by parts there is obtained

$$R = p_o(1-\tau_o)^{\beta+1}(l_1+l_2)-2p_1 l + (\beta+1)p_o\int_0^{\tau_o}(1-\tau)^{\beta} d\tau(y_{\theta=\frac{\pi}{2}} - y_{\theta=-\frac{\pi}{2}})$$

Substitute, in this expression, the variable σ; from the relation between τ and σ (formulas (91) and (93)),

$$(1-\tau)^{\beta}d\tau = -2\tau d\sigma = -\frac{8}{(C_1e^{\sigma} + C_2e^{-\sigma})^2}d\sigma$$

and the limits of integration with respect to σ are ∞ and 0; moreover it may be noted that the first two terms in the formula for R cancel, since $l_1 + l_2 = 2l$, $p_o(1-\tau_o)^{\beta+1} = p_1$. Hence

$$R=8(\beta+1)p_0\int_0^{\infty}(y_{\theta=\frac{\pi}{2}}-y_{\theta=-\frac{\pi}{2}})\frac{d\sigma}{(C_1e^{\sigma}+C_2e^{-\sigma})^2},$$

or, integrating by parts,

$$\frac{C_1R}{4(\beta+1)p_0}=-\Big/_{0}^{\infty}(y_{\theta=\frac{\pi}{2}}-y_{\theta=-\frac{\pi}{2}})\frac{1}{C_1e^{2\sigma}+C_2}+$$

$$+\int_0^{\infty}\left\{\left(\frac{\partial y}{\partial\sigma}\right)_{\theta=\frac{\pi}{2}}-\left(\frac{\partial y}{\partial\sigma}\right)_{\theta=-\frac{\pi}{2}}\right\}\frac{d\sigma}{C_1e^{2\sigma}+C_2}.$$

Substitution gives the result $2l/(C_1+C_2)$.

With regard to the remaining integral, by equation (94), if $\theta = \pi/2$ and it is remembered here that $\psi = 0$

$$2\sqrt{2\alpha}\left(\frac{\partial y}{\partial\sigma}\right)_{\theta=\frac{\pi}{2}}=(C_1e^{\sigma}+C_2e^{-\sigma})\left(\frac{d\varphi}{d\sigma}\right)_{\theta=\frac{\pi}{2}};$$

For $\theta = -\pi/2$, $\psi = 0$ there is obtained

$$2\sqrt{2\alpha}\left(\frac{\partial y}{\partial\sigma}\right)_{\theta=-\frac{\pi}{2}}=-(C_1e^{\sigma}+C_2e^{-\sigma})\left(\frac{d\varphi}{d\sigma}\right)_{\theta=-\frac{\pi}{2}}.$$

Making use of these formulas and integrating again by parts, in the expression for R, reduces it to the form

$$\frac{C_1R\sqrt{2\alpha}}{2(\beta+1)p_0}=\frac{4l\sqrt{2\alpha}}{C_1+C_2}+\Big/_{0}^{\infty}(\varphi_{\theta=\frac{\pi}{2}}+\varphi_{\theta=-\frac{\pi}{2}})e^{-\sigma}+$$

$$+\int_0^{\infty}(\varphi_{\theta=\frac{\pi}{2}}+\varphi_{\theta=-\frac{\pi}{2}})e^{-\sigma}d\sigma.$$

From equations (95) there is

$$\varphi_{\theta=\frac{\pi}{2}}=\frac{k\cos^4\lambda}{\left(\sin\lambda+\frac{e^{\sigma}+e^{-\sigma}}{2}\right)^2},\quad \varphi_{\theta=-\frac{\pi}{2}}=\frac{k\cos^4\lambda}{\left(\sin\lambda-\frac{e^{\sigma}+e^{-\sigma}}{2}\right)^2};$$

and, therefore, by substitution, and introduction in the integral of the variable $\xi = e^{-\sigma}$, there is obtained:

$$\frac{C_1 R\sqrt{2\alpha}}{2(\beta+1)p_0} = \frac{4l\sqrt{2\alpha}}{C_1+C_2} - 2k(1+\sin^2\lambda) + 4k\cos^4\lambda \int_0^1 \frac{\xi^2 d\xi}{(1+\xi^2+2\xi\sin\lambda)^2} +$$

$$+ 4k\cos^4\lambda \int_0^1 \frac{\xi^2 d\xi}{(1+\xi^2-2\xi\sin\lambda)^2}\,.$$

These definite integrals have, respectively, the values

$$\frac{1}{2\cos^2\lambda}\left\{\frac{\sin\lambda-\cos 2\lambda}{2+2\sin\lambda} - \sin\lambda + \frac{1}{\cos\lambda}\left(\frac{\pi}{4}-\frac{\lambda}{2}\right)\right\}$$

and

$$\frac{1}{2\cos^2\lambda}\left\{-\frac{\sin\lambda+\cos 2\lambda}{2-2\sin\lambda} + \sin\lambda + \frac{1}{\cos\lambda}\left(\frac{\pi}{4}+\frac{\lambda}{2}\right)\right\},$$

and hence, on adding, give

$$\frac{1}{2\cos^2\lambda}\left(-1+\frac{\pi}{2\cos\lambda}\right);$$

Substitution of this expression in the formula for R yields

$$\frac{C_1 R\sqrt{2\alpha}}{2(\beta+1)p_0} = \frac{4l\sqrt{2\alpha}}{C_1+C_2} - 4k + \pi k\cos\lambda.$$

Since

$$C_1+C_2 = \frac{2}{\sqrt{\tau_0}}\,,\quad 2l\,.\sqrt{2\tau_0\alpha} = 4k + \pi k\cos\lambda(1-\tau_0)^{-\beta},\ (\text{cf. } (97)),$$

$$C_1 = \frac{1+(1-\tau_0)^{-\beta}}{\sqrt{\tau_0}}\,,$$

therefore

$$R = \frac{2\pi}{\sqrt{2\alpha}}\,k\cos\lambda\sqrt{\tau_0}(\beta+1)p_0. \qquad (98)$$

From the formula just given for $2l$:

$$\frac{k}{\sqrt{2\alpha}} = \frac{2l\sqrt{\tau_0}}{4+\pi(1-\tau_0)^{-\beta}}\,;$$

Moreover

$$p_o = K\rho_o^{\gamma}, \quad (\beta+1)p_o = \frac{\gamma K \rho_o^{\gamma}}{\gamma-1} = \alpha\rho_o = \alpha\rho_1 \ (1-\tau_o)^{-\beta}$$

Since from the definition of the constant α (See pt. I.)

$$\alpha = \frac{K\gamma\rho_o^{\gamma-1}}{\gamma-1}$$

and the density at the jet surface ρ_1, equal to the density of the distant regions of the flow, is connected with the density at the critical point ρ_o by the formula

$$\rho_1 = \rho_o(1-\tau_o)^{\beta}$$

Finally by taking into account the equation $\sqrt{2\alpha\tau_o} = v_o$, the velocity at the jet surface and at the infinitely distant points of the moving gas mass, there is obtained from (98):

$$R = \frac{\pi\cos\lambda}{4(1-\tau_o)^{\beta} + \pi\cos\lambda} \, 2l v_o^{2}\rho_1$$

This formula for $\beta = 0$ passes over, as it should, into the formula of Lord Rayleigh for the flow of an incompressible liquid, and for $\lambda = 0$ gives the approximate solution of the problem of the pressure of a symmetrical gas flow on a plate. Computing the coefficient $\frac{\pi\cos\lambda}{4(1-\tau_o)^{\beta}+\pi\cos\lambda}$ for the values of s_o assumed at the beginning of this part and for which this approximate method is applicable and considering only the case of symmetrical flow leads to the following result: <u>For a change in s_o from 0 to $\frac{1}{12\beta}$ and flow velocity from 0 to 136 meters per second the coefficient $\frac{\pi}{4(1-\tau_o)^{\beta}+\pi}$ fluctuates within the limits</u>

0.440 to 0.460, the increase at the end being greater than at the beginning. Thus for $s_o = \frac{1}{24\beta}$ and $v_o = 96$ meters per second it still equals only 0.449. Hence for not very large velocities the law of the proportionality of the resistance to the square of the velocity is found to be almost exact.

SUPPLEMENTARY REMARKS

1.

Part II: It is of interest to note that the function y_n will always have real roots within the limits of the variation of τ from its critical value $\frac{1}{2\beta+1}$ to 1, provided n is sufficiently large. Thus for functions with integral n it is true for $n > 1$. The number of roots increases infinitely with n. These results are obtained from Porter's article (reference 12). It is readily concluded that the solution of the problems on the flow of a gas out of a vessel and the resistance of a moving plate in air, given in parts III and IV, are not applicable outside of the limits indicated in this paper because of the divergence of the series expressing the stream function and velocity potential.

2.

Part V: The expression $K = [1 - (2\beta+1)\tau](1-\tau)^{-2\beta-1}$ which, in presenting the "approximate method," was accepted as equal to unity actually will be equal to unity in two cases:

1. If $\beta = 0$: This is the case of the motion of an incompressible liquid, since the formula for the density $\rho = \rho_o(1-\tau)^\beta$ reduces to the equation ρ = constant.

2. If $\beta = -\frac{1}{2}$: In this case $p = k\rho^{1+\frac{1}{\beta}} = k/\rho$. If the moving matter is an ideal gas, then in order that this condition may be satisfied, it is necessary, in some manner, to remove the heat from the flowing mass of gas. To create

such a state of motion of the gas is not actually possible. The problem, however, arrived at on setting $\beta = -\frac{1}{2}$ is of interest from quite another viewpoint.

The initial equations (7) of part I is considered. By substitution in them for τ its value $\frac{1}{2\alpha}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right]$ 2α equal to -1 is taken and, for briefness, the derivatives of φ are denoted with respect to x and y by p and q, respectively. The equations then become

$$\frac{p}{\sqrt{1+p^2+q^2}} = \frac{\partial\psi}{\partial y}, \qquad \frac{q}{\sqrt{1+p^2+q^2}} = -\frac{\partial\psi}{\partial x}$$

$$\frac{p\,dy - q\,dx}{\sqrt{1+p^2+q^2}} = d\psi$$

Hence it is clear that if we put $\varphi = z$, then x, y, z will be the rectangular coordinates of the points of a minimal surface.

Formulas (91) to (95) of part V, on substituting $-u$ for τ, lead to the following relations:

$$\frac{1}{\sqrt{u}} = C_1 e^{\sigma} + C_2 e^{-\sigma}, \qquad \sqrt{\frac{1+u}{u}} = C_1 e^{\sigma} - C_2 e^{-\sigma}$$

$$C_1 + C_2 = \frac{1}{\sqrt{u_0}}, \quad C_1 - C_2 = \sqrt{\frac{1+u_0}{u_0}}$$

where the arbitrary constants are given somewhat different values from those in the formulas of V.

If $\sigma + i\theta = t$, $\sigma - i\theta = t_1$, then

$$z + i\psi = f(t)$$

$$x + iy = C_1 \int e^{-t} f'(t)\,dt + C_2 \int e^{-t_1} f_1'(t_1)\,dt_1$$

where

$$u = p^2 + q^2, \qquad \theta = \operatorname{arctg} \frac{q}{p}$$

For the square of the linear element of the surface there is found the expression

$$ds^2 = (C_1 e^{\sigma} - C_2 e^{-\sigma})^2 (dz^2 + d\Psi^2) = \frac{1+p^2+q^2}{p^2+q^2} (dz^2 + d\Psi^2)$$

If the xy plane is horizontal the curves z = constant will be the horizontals of the surface; Ψ = constant are their orthogonal trajectories.

From the foregoing equations minimal surfaces of various shapes may be derived.

1. Setting

$$f(t) = e^{nt}$$

yields

$$x + iy = \frac{nC_1}{n+1} (z+i\Psi)^{\frac{n+1}{n}} + \frac{nC_2}{n-1} (z-i\Psi)^{\frac{n-1}{n}}$$

For n rational various shapes of algebraic surfaces are thus obtained. An exception is the case $n = 1$, the surface then being transcendental.

Setting $f(t) = At$ gives for real A the catenoid and for A the helicoid.

2. A second group of minimal surfaces obtained from the above formulas is of much greater interest. With the aid of the latter the minimal surface described within a certain given polygonal contour may be sought. The latter should consist of horizontal and vertical straight segments (the xy plane as before is taken to be some horizontal plane). On setting for simplicity $u_0 = \infty$ and hence $C_1 = - C_2 = 1/2$ in the above formulas the following is noted: On each horizontal segment of the boundary contour

there will evidently be z = constant and θ = constant; if, however, the segment under consideration is vertical then on it ψ = constant, $p^2 + q^2 = \infty$ and $\sigma = 0$. Thus the regions $z + i\psi$ and $\sigma + i\theta$ will be bounded by straight lines. By finding the conformal transformation of these regions on the upper half plane of the auxiliary complex variable s, by the known method, the problem to the effecting of quadratures is reduced. As a very simple example, the surface described in a pentagonal contour of the following shape is obtained; one of its sides is the segment of the y axis bisected by the origin of coordinates; from the end of this segment are drawn two equal sides parallel to the z axis; from the ends of the latter two infinite lines parallel to the x axis are drawn thus completing the contour. This surface is expressed by the following equations:

$$\sin \frac{2y\sqrt{1-k^2}}{a} = \frac{2\,\mathrm{sn}\,\frac{\psi}{a}\,\mathrm{cn}\,\frac{\psi}{a}\,\mathrm{dn}\,\frac{iz}{a}}{1-k^2\mathrm{sn}^2\,\frac{\psi}{a}\,\mathrm{sn}^2\frac{iz}{a}}$$

$$\sinh \frac{2x\sqrt{1-k^2}}{a} = -\frac{2ki\,\mathrm{sn}\,\frac{iz}{a}\,\mathrm{dn}\,\frac{iz}{a}\,\mathrm{cn}\,\frac{\psi}{a}}{1-k^2\mathrm{sn}^2\,\frac{\psi}{a}\,\mathrm{sn}^2\,\frac{iz}{a}}$$

They are readily obtained with the aid of the preceding general formulas if

$$\sigma + i\theta = \lg \frac{k\sqrt{1-s^2}+\sqrt{1-k^2s^2}}{\sqrt{1-k^2}}$$

$$z + i\psi = ai\int_0^s \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}$$

In conclusion, it may be noted that the given conditions for the surface may be somewhat varied. Thus, among the conditions, the requirement that one of the horizontals be a line of curvature of the surface may be included. The plane of this horizontal will then intersect the required surface at a constant angle and, therefore, it will be found that, for a certain given value of z, $p^2 + q^2$ = constant and hence σ = constant. In exactly the same way, if it is known that one of the curves, ψ = constant, is a plane curve, then, as is easily shown, along this curve the angle θ will be constant.

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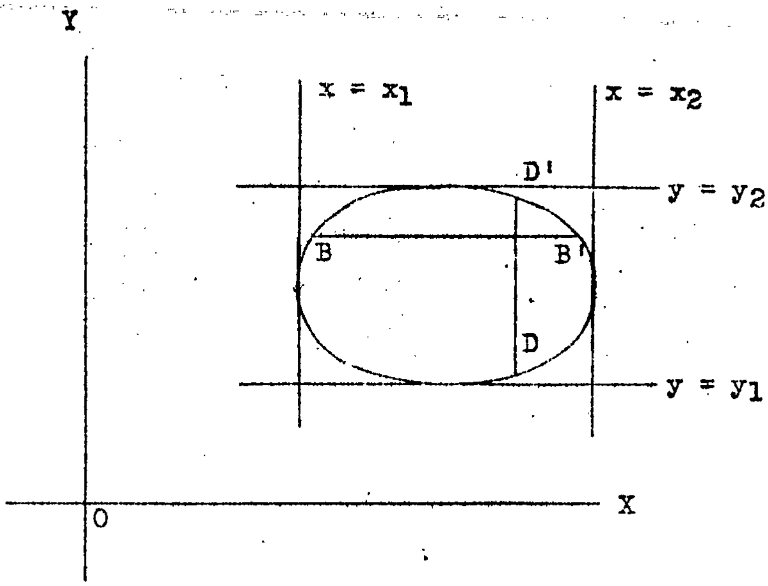


Figure 1.

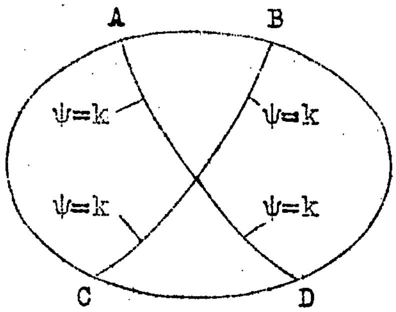


Figure 2.

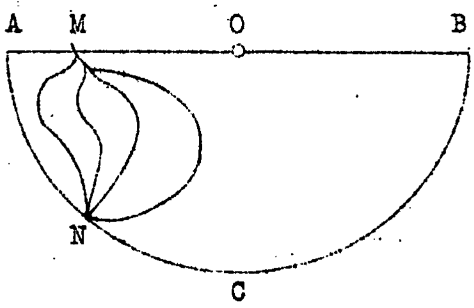
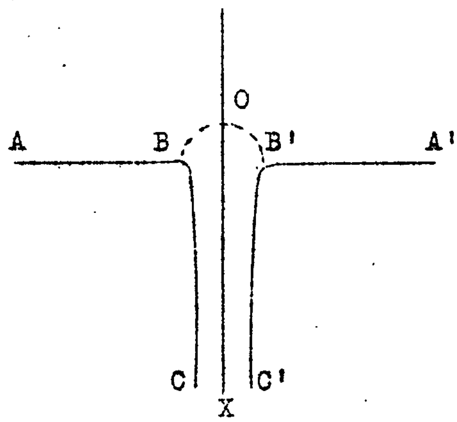
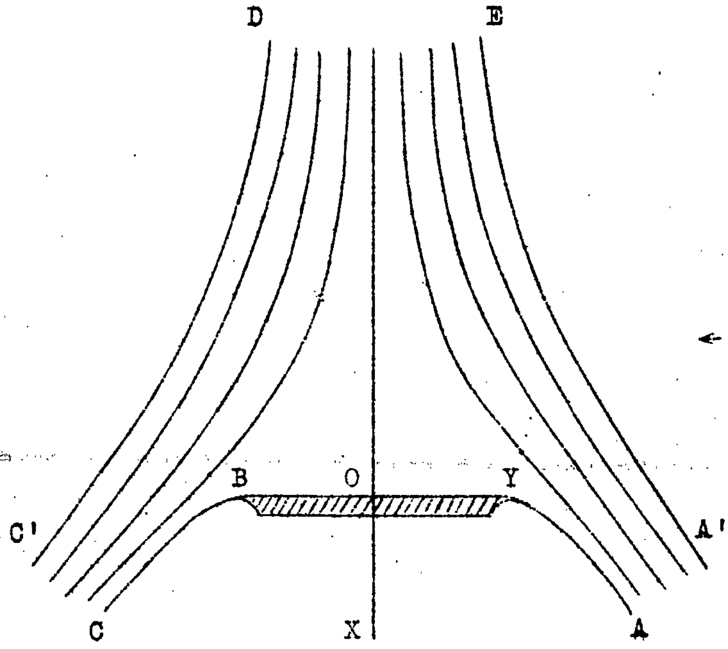


Figure 3.

Figure 4. →





← Figure 5.

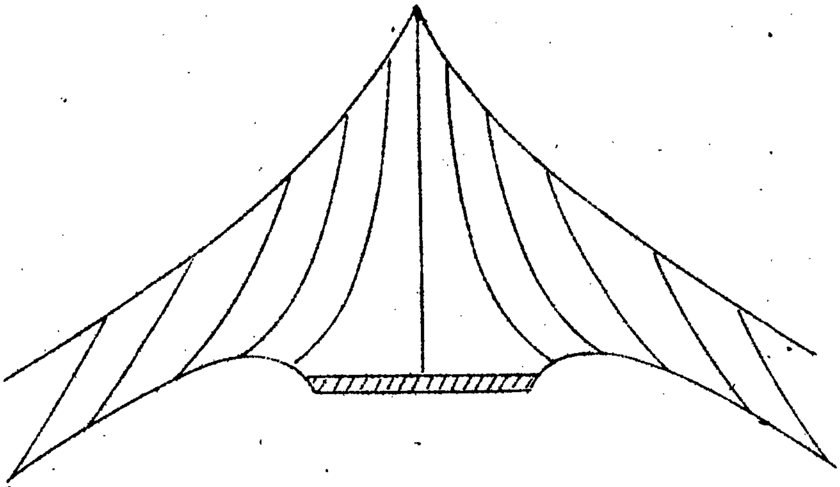


Figure 6.

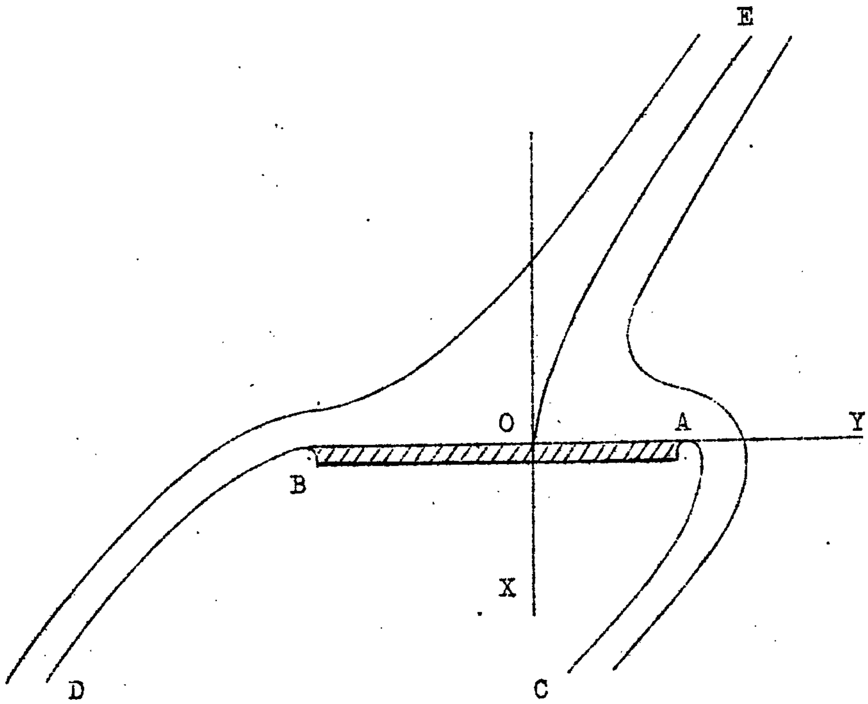


Figure 7.